INTEGRATION AND DERIVATIVE SEQUENCES FOR PELL AND PELL-LUCAS POLYNOMIALS

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1. INTRODUCTION

Previously in [1] and [2], in which integration and first derivative sequences for Fibonacci and Lucas polynomials were introduced, it was suggested that these investigations could be extended to Pell and Pell-Lucas polynomials. Here, we explore some of their basic features in outline to obtain the flavor of their substance. Further details may be found in [5], with some variation in notation.

Pell polynomials $P_n(x)$ are defined by the recurrence relation

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad P_0(x) = 0, P_1(x) = 1, \tag{1.1}$$

while the associated *Pell-Lucas polynomials* $Q_n(x)$ are defined by

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \quad Q_0(x) = 2, Q_1(x) = 2x. \tag{1.2}$$

Standard procedures readily lead to the *Binet forms*

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{2\Delta(x)} \tag{1.3}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x), \tag{1.4}$$

where

$$\Delta(x) = \sqrt{x^2+1}, \ \alpha(x) = x + \Delta(x), \ \beta(x) = x - \Delta(x). \tag{1.5}$$

Properties of $P_n(x)$ and $Q_n(x)$ are given in [3] and [5].

Substitution of $x = 1$ in (1.1) and (1.2) leads to the corresponding *Pell numbers* $P_n = P_n(1)$ and *Pell-Lucas numbers* $Q_n = Q_n(1)$. For reference, we tabulate some values of P_n and Q_n:

n	0	1	2	3	4	5	6	7	8	$\cdots$
P_n	0	1	2	5	12	29	70	169	408	$\cdots$
Q_n	2	2	6	14	34	82	198	478	1154	$\cdots$

(1.6)

All the Q_n are even numbers, as is manifest from (1.2). The P_n are alternately odd and even.

2. PROPERTIES OF DERIVATIVE SEQUENCES

Using known [3] summation formulas for $P_n(x)$ and $Q_n(x)$, we derive the *first derivative Pell sequence* $\{P_n'(x)\}$ given by

$$P_n'(x) = 2\sum_{m=0}^{\left[\frac{n-1}{2}\right]}(n-2m-1)\binom{n-m-1}{m}(2x)^{n-2m-2} \quad (n \geq 1) \tag{2.1}$$

and the *first derivative Pell-Lucas sequence* $\{Q_n'(x)\}$ for which

$$Q_n'(x) = 2n\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n-m-1}{m}(2x)^{n-2m-1} \quad (n \geq 1), \tag{2.2}$$

where the dash denotes differentiation with respect to x and the symbol $[\cdot]$ represents the greatest integer function.

From (1.1) and (1.2), we must have

$$P_0'(x) = 0 \quad \text{and} \quad Q_0'(x) = 0. \tag{2.3}$$

Expressions (2.1) and (2.2) yield the first few polynomials $P_n'(x)$ and $Q_n'(x)$ [5]:

$$\begin{array}{ll}
P_1'(x) = 0 & Q_1'(x) = 2 \\
P_2'(x) = 2 & Q_2'(x) = 8x \\
P_3'(x) = 8x & Q_3'(x) = 24x^2 + 6 \\
P_4'(x) = 24x^2 + 4 & Q_4'(x) = 64x^3 + 32x \\
P_5'(x) = 64x^3 + 24x & Q_5'(x) = 160x^4 + 120x^2 + 10 \\
P_6'(x) = 160x^4 + 96x^2 + 6 & Q_6'(x) = 384x^5 + 384x^3 + 72x \\
P_7'(x) = 384x^5 + 320x^3 + 48x & Q_7'(x) = 896x^6 + 1120x^4 + 336x^2 + 14 \\
P_8'(x) = 896x^6 + 960x^4 + 240x^2 + 8 & Q_8'(x) = 2048x^7 + 3072x^5 + 1280x^3 + 128x\,.
\end{array} \tag{2.4}$$

Putting $x = 1$ in (2.4), we derive the corresponding first derivative Pell sequence numbers $\{P_n'\} = \{P_n'(1)\}$ and first derivative Pell-Lucas sequence numbers $\{Q_n'\} = \{Q_n'(1)\}$, tabulated thus [5]:

n	0	1	2	3	4	5	6	7	8	...
P_n'	0	0	2	8	28	88	262	752	2104	⋯
Q_n'	0	2	8	30	96	290	840	2366	6528	⋯

(2.5)

All the numbers P_n' and Q_n' are even, by virtue of the factor 2 in (2.1) and (2.2).

Elementary calculations using (1.5) produce

$$\alpha'(x) = \frac{\alpha(x)}{\Delta(x)}, \tag{2.6}$$

$$\beta'(x) = -\frac{\beta(x)}{\Delta(x)}, \tag{2.7}$$

$$\{\alpha^n(x)\}' = \frac{n\alpha^n(x)}{\Delta(x)}, \tag{2.8}$$

$$\{\beta^n(x)\}' = -\frac{n\beta^n(x)}{\Delta(x)}, \tag{2.9}$$

whence we derive, after a little calculation using (1.3) and (1.4),

$$P_n'(x) = \frac{nQ_n(x) - 2xP_n(x)}{2\Delta^2(x)} \tag{2.10}$$

and

$$Q_n'(x) = 2nP_n(x). \tag{2.11}$$

Taken in conjunction with (1.3) and (1.4), equations (2.10) and (2.11) allow us to express $P_n'(x)$ and $Q_n'(x)$ in their Binet forms.

Substituting $x = 1$ in (2.10) and (2.11), we have immediately

$$P_n' = \frac{nQ_n - 2P_n}{4} \tag{2.12}$$

and

$$Q_n' = 2nP_n. \tag{2.13}$$

For example, $P_6' = 262 = \frac{6\cdot198-2\cdot70}{4} = \frac{6Q_6-2P_6}{4}$ by (2.12) and (1.6).

Other basic results are [5]:

$$\left.\begin{aligned} P_n' &= 2P_{n-1}' + P_{n-2}' + 2P_{n-1} \\ Q_n' &= 2Q_{n-1}' + Q_{n-2}' + 2Q_{n-1} \end{aligned}\right\} \textit{recurrence relations}, \tag{2.14}$$

$$P_{n+1}' + P_{n-1}' = Q_n', \tag{2.16}$$

$$Q_{n+1}' + Q_{n-1}' = 2nQ_n + 4P_n, \tag{2.17}$$

$$P_{n+1}'P_{n-1}' - (P_n')^2 = \frac{8n^2(-1)^{n+1} + 4(-1)^n - Q_n^2}{16} \quad (\textit{Simson's formula}), \tag{2.18}$$

and

$$Q_{n+1}'Q_{n-1}' - (Q_n')^2 = 4\{(-1)^n(n^2-1) - P_n^2\} \quad (\textit{Simson's formula}). \tag{2.19}$$

To obtain these results, we use (2.12) and (2.13) as well as properties of P_n and Q_n (1.6). Proof of Simson's formula (2.18) requires much careful calculation though (2.19) follows readily from (2.13) and Simson's formula for P_n. One may note *en passant* that (2.16) is analogous to the well-known relations between P_n and Q_n, and F_n and L_n (Fibonacci and Lucas numbers).

Numerical illustrations of (2.14), (2.17), and (2.18) are, by (1.6) and (2.5), respectively,

$$n = 5: \quad 2P_4' + P_3' + 2P_4 = 56 + 8 + 24 = 88 = P_5',$$

$$n = 5: \quad Q_6' + Q_4' = 840 + 96 = 936 = 10\cdot82 + 4\cdot29 = 10Q_5 + 4P_5,$$

$$n = 5: \quad \begin{cases} P_6'P_4' - (P_5')^2 = 262\cdot28 - 88^2 = -408, \\ \dfrac{8\cdot5^2(-1)^{5+1} + 4(-1)^5 - Q_5^2}{16} = \dfrac{200-4-6724}{16} = \dfrac{-6528}{16} = -408. \end{cases}$$

Analogues of Simson's formulas (2.18) and (2.19) can be obtained for $P_n'(x)$ and $Q_n'(x)$.

3. INTEGRATION SEQUENCES

Consider, in a new notation, the integrals [5]

$$'P_n(x) = \int_0^x P_n(s)\,ds \tag{3.1}$$

and

$$'Q_n(x) = \int_0^x Q_n(s)\,ds, \tag{3.2}$$

where the pre-symbol dash represents integration.

Using the summation formulas for $P_n(x)$ and $Q_n(x)$ [3], we readily obtain

$$'P_n(x) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} \frac{2^{n-2m-1}}{n-2m}\binom{n-1-m}{m} x^{n-2m} \quad (n \ge 1) \tag{3.3}$$

and

$$'Q_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{n2^{n-2m}}{(n-m)(n-2m+1)}\binom{n-m}{m} x^{n-2m+1} \quad (n \ge 1). \tag{3.4}$$

From (1.1), (1.2), (3.1), and (3.2), we deduce that

$$'P_0(x) = 0, \quad 'Q_0(x) = 2x. \tag{3.5}$$

Sequences $\{'P_n(x)\}$ and $\{'Q_n(x)\}$ may be called the *Pell integration sequence* and the *Pell-Lucas integration sequence*, respectively. Their first few expressions, obtained from (3.3) and (3.4), are [5]:

$$\begin{array}{ll}
'P_1(x) = x & 'Q_1(x) = x^2 \\
'P_2(x) = x^2 & 'Q_2(x) = \frac{4}{3}x^3 + 2x \\
'P_3(x) = \frac{4}{3}x^3 + x & 'Q_3(x) = 2x^4 + 3x^2 \\
'P_4(x) = 2x^4 + 2x^2 & 'Q_4(x) = \frac{16}{5}x^5 + \frac{16}{3}x^3 + 2x \\
'P_5(x) = \frac{16}{5}x^5 + 4x^3 + x & 'Q_5(x) = \frac{16}{3}x^6 + 10x^4 + 5x^2 \\
'P_6(x) = \frac{16}{3}x^6 + 8x^4 + 3x^2 & 'Q_6(x) = \frac{64}{7}x^7 + \frac{96}{5}x^5 + 12x^3 + 2x \\
'P_7(x) = \frac{64}{7}x^7 + 16x^5 + 8x^3 + x & 'Q_7(x) = 16x^8 + \frac{112}{3}x^6 + 28x^4 + 7x^2 \\
'P_8(x) = 16x^8 + 32x^6 + 20x^4 + 4x^2 & 'Q_8(x) = \frac{256}{9}x^9 + \frac{512}{7}x^7 + 64x^5 + \frac{64}{3}x^3 + 2x.
\end{array} \tag{3.6}$$

Putting $x = 1$ in (3.6), we obtain the Pell integration sequence numbers $\{'P_n(1)\} = \{'P_n\}$, and the Pell-Lucas integration sequence numbers $\{'Q_n(1)\} = \{'Q_n\}$, respectively, of which the first few members are [5]:

n	0	1	2	3	4	5	6	7	8	...
$'P_n$	0	1	1	$\frac{7}{3}$	4	$\frac{41}{5}$	$\frac{49}{3}$	$\frac{239}{7}$	72	...
$'Q_n$	2	1	$\frac{10}{3}$	5	$\frac{158}{15}$	$\frac{61}{3}$	$\frac{1482}{35}$	$\frac{265}{3}$	$\frac{11902}{63}$	...

(3.7)

Two elementary properties of $\{'P_n\}$ and $\{'Q_n\}$ are

$$'P_n = \begin{cases} \frac{Q_n}{2n} & (n>0,\ \text{odd}) \\ \frac{Q_n-2}{2n} & (n>0,\ \text{even}) \end{cases} \tag{3.8}$$

and

$$'Q_n = \begin{cases} \frac{2n(2P_n-1)-Q_n}{n^2-1} & (n>1,\ \text{odd}) \\ \frac{4nP_n-Q_n}{n^2-1} & (n>1,\ \text{even}). \end{cases} \tag{3.9}$$

Proofs of these [5] are lengthy but of a relatively elementary nature and are omitted to conserve space. The procedure is to begin with (3.1), (3.2), then integrate with the aid of (1.3)-(1.5), and eventually set $x = 1$, taking into account the values of $P_n(0)$ and $Q_n(0)$ for n even and n odd.

Complicated Binet forms of $'P_n(x)$, $'Q_n(x)$, $'P_n$, and $'Q_n$ are obtainable on applying the corresponding Binet forms for the undashed symbols from (1.3) and (1.4).

From (3.6) and (3.7), we may obtain

$$'P_{n+1} + 'P_{n-1} = 'Q_n \tag{3.10}$$

and

$$'P_{n+1} - 'P_{n-1} = \frac{Q_n - 'Q_n}{n}. \tag{3.11}$$

Once again, it is worth commenting on the fundamental nature of property (3.10) [cf. (2.16)].

Numerical illustrations of (3.10) and (3.11) are, respectively,

$$n=4:\quad 'P_5 + 'P_3 = \frac{41}{5} + \frac{7}{3} = \frac{158}{15} = 'Q_4,$$

$$n=6:\quad 'P_7 - 'P_5 = \frac{239}{7} - \frac{41}{5} = \frac{908}{35} = \frac{198 - \frac{1482}{35}}{6} = \frac{Q_6 - 'Q_6}{6}.$$

The Simson formula analogue for $'P_n$ takes two forms, depending on whether n is odd or even. From (3.8), and invoking the use of Simson's formula for Q_n, we obtain

$$'P_{n+1}\,'P_{n-1} - ('P_n)^2 = \begin{cases} \frac{8(-1)^{n+1}n^2+4n^2+Q_n^2-16n^2P_n}{4n^2(n^2-1)} & (n\ \text{odd}) \\ \frac{8(-1)^{n+1}n^2+Q_n^2+4(n^2-1)(Q_n-1)}{4n^2(n^2-1)} & (n\ \text{even}). \end{cases} \tag{3.12}$$

As an example, when $n = 5$, both sides of (3.12) equal $-\frac{143}{75}$, whereas, if $n = 4$, both sides reduce to $\frac{47}{15}$.

From (3.9), a Simson formula analogue for $'Q_n$ is clearly obtainable but its form is left to the curiosity of the reader. Corresponding analogues also exist for $'P_n(x)$ and $'Q_n(x)$.

To check the consistency of the results, one might establish that $'(P_n'(x)) = ('P_n(x))' = P_n(x)$ and similarly for $Q_n(x)$.

4. CONCLUDING REMARKS

Extensions:

Two observations on the foregoing material are relevant:

(i) clearly, the procedures for obtaining integration and first derivative sequences for Fibonacci and Lucas polynomials as in [1] and [2], and for Pell and Pell-Lucas polynomials as herein, can be made more general to embody *multiple integration sequences* and n^{th}*-order derivative sequences*, and

(ii) the ideas delineated here are applicable to the generalized recurrence-generated polynomials for which the coefficient $2x$ in (1.1) and (1.2) is replaced by kx, with appropriate initial conditions.

Simson v Simpson:

Occurrences of analogues to Simson's original formula in 1753 for Fibonacci numbers [4], and the frequent misspellings of Simson's name, prompt us to offer a brief, if only peripherally relevant, historical explanation to clarify the situation. The formula is due to the distinguished Scot, Robert Simson (1687-1768), who was also the author of a highly successful text-book on Euclidean geometry. He is not to be confused with his able contemporary English mathematician, Thomas Simpson (1710-1761), whose name is associated with the rule for approximate quadratures by means of parabolic arcs. Our man is Robert Simson.

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