MODIFIED NUMERICAL TRIANGLE AND THE FIBONACCI SEQUENCE

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1. INTRODUCTION

A study of the number $(1+\sqrt{5})/2 \cong 1.618\ldots$ and of its Fibonacci derivation has received considerable attention not only in the field of pure mathematics but also in mathematical modeling and analysis of such physical plants as cascades of two-ports, hot mill metallurgical processes, multicomponent rectifications in distillation column, reactions in stirred tank reactors and batch reactors [6]. As the mathematical basis for solutions of various problems in these systems serves usually the theory of recurrence equations ([2], [3]) of the Fibonacci sequence and their generalizations ([1], [4]). Many problems concerning a variety of generalizations of the Fibonacci sequence have appeared, primarily in *The Fibonacci Quarterly*, in recent years.

We shall be concerned in this paper with the Fibonacci sequence introduced via a modified numerical triangle (MNT). We shall involve a generalized Pascal triangle (GGPT) and "shifted" form of the MNT (SMNT) and show how the MNT results from a suitable superposition of the generalized and shifted triangles. We shall also prove that a transfer ratio T_k $(k=0,1,2,\ldots,n)$ of the output to input-signal along an electrical ladder network is determined by polynomials with coefficients belonging to the MNT.

2. THE MODIFIED NUMERICAL TRIANGLE

The MNT is defined here in connection with studies of distributions of voltages and currents along an electrical ladder network with n identical interacting cells [5]. One elementary section of such structures is characterized by a parameter x determined by the product of impedance of a longitudinal branch and admittance of a transversal branch.

The transfer ratio T_k $(k=0,1,2,\ldots,n)$ of the output- to input-signal (voltage or current) along the network (Fig. 1) is determined by polynomials in x of the corresponding degree. It can be determined from a solution of the following recurrence equation,

$$a_{k+1}-(2+x)a_k+a_{k-1}=0 \tag{1a}$$

with

$$a_1=(1+x)a_0, \tag{1b}$$

where a_0 denotes a known signal at the input port of the first cell and a_k is the corresponding signal at the k-port of the network (e.g., $a_k=V_k$ as shown in Fig. 1).

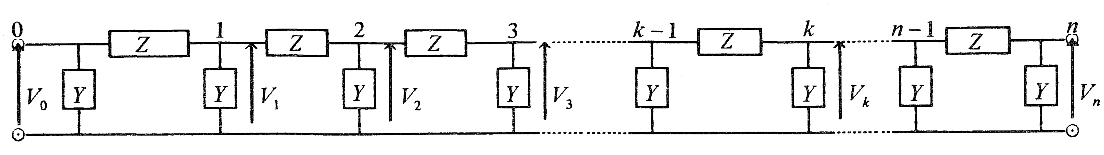

Figure 1

The ratio T_k follows from the relation

$$T_k = \frac{a_k}{a_0}, \quad k = 0, 1, 2, \ldots, n. \tag{2}$$

It is easy to see that T_k is determined by a polynomial in x of the k^{th} degree, so we can write

$$T_k = \sum_{m=0}^{k} p_{k,m} x^m, \quad k = 0, 1, 2, \ldots, n. \tag{3}$$

From direct inspection of the above expression, we have that

$$\begin{aligned}
&T_0(x) = 1, \\
&T_1(x) = 1 + x, \\
&T_2(x) = 1 + 3x + x^2, \\
&T_3(x) = 1 + 6x + 5x^2 + x^3, \\
&T_4(x) = 1 + 10x + 15x^2 + 7x^3 + x^4, \\
&T_5(x) = 1 + 15x + 35x^2 + 28x^3 + 9x^4 + x^5, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{4}$$

The polynomial coefficients

$$p_{k,m} = \frac{1}{m!} \left. \frac{\partial^m T_k(x)}{\partial^m x} \right|_{x=0} \tag{5}$$

belong to the MNT that takes the following form:

k	MNT					
0	1					
1	1	1				
2	1	3	1			
3	1	6	5	1		
4	1	10	15	7	1	
5	1	15	35	28	9	1
...						

It must be noted that from the MNT it is easy to obtain the expression of the polynomial $T_k(x)$ for small values of k. To determine $T_k(x)$ for large values of k, we can use formulas (1) and (2).

Observe that formally the MNT is apparently similar to the classic Pascal Triangle. Note that the MNT coefficients cannot be evaluated directly by applying the rule corresponding to the classic Pascal Triangle. On the other hand, it is possible, by some appropriate modification of the Pascal Triangle, to establish a suitable recurrence rule for constructing the MNT coefficients. We will present a solution to this problem in the next section.

3. THE GENERALIZED AND SHIFTED TRIANGLES AND THEIR LINKS WITH THE MNT

By a slight modification of the MNT we can establish the so-called shifted modified numerical triangle (SMNT). We draw SMNT from MNT by shifting its rows and columns by two places in the bottom and then annihilating all coefficients in the first column. If we denote by $s_{k,m}$ a coefficient for a node (k, m), $k = 0, 1, 2, \ldots, n$ and $m = 0, 1, \ldots, k$, then the corresponding formula takes the following form:

$$s_{k,m} = \begin{cases} 0 & \text{for } m = 0, \\ p_{k-2,m} & \text{for } 1 \le m \le k-2, \\ 0 & \text{for } m > k-2. \end{cases} \tag{6}$$

The resulting SMNT is demonstrated by the following construction:

k	SMNT							
0	0							
1	0	0						
2	0	0	0					
3	0	1	0	0				
4	0	3	1	0	0			
5	0	6	5	1	0	0		
6	0	10	15	7	1	0	0	
7	0	15	35	28	9	1	0	0

....

Applying the above rule, for instance for $s_{5,2}$, we obtain

$$s_{5,2} = p_{3,2} = 5.$$

The GGPT is constructed similarly to the usual Pascal Triangle (PT), with only two modifications. First, in evaluating a given node numerical element in the GGPT, its upper right-hand side node element is taken twice and the upper left-hand node coefficient is taken in the same way as in the classic PT. Second, before performing calculations for node coefficients in the $(k + 1)$ row of the GGPT we must subtract the k^{th} row of the SMNT from the k^{th} row of the GPT. If we denote by $g_{k,m}$ the GGPT coefficient corresponding to the $(k, m)^{\text{th}}$ node, then the following rule,

$$g_{k,m} = g_{k-1,m-1} - s_{k-1,m-1} + 2(g_{k-1,m} - s_{k-1,m}), \tag{7}$$

holds for $k = 0, 1, 2, \ldots, n$ and $m = 0, 1, \ldots, k$ with $g_{k,0} = 1$ and $g_{k-1,m} = 0$ for $m > k-1$ and $g_{k-1,m-1} = 0$ for $m - 1 < 0$. The above rule is illustrated by the following representation of the GGPT:

k	GGPT					
0	1					
1	1	1				
2	1	3	1			
3	1	7	5	1		
4	1	13	16	7	1	
5	1	21	40	29	9	1

....

Now we can show a link between the MNT and the generalized and shifted triangles. Applying successively, row-by-row, the rules corresponding to the GGPT and SMNT for the MNT and comparing coefficients that correspond to a given node in all three triangles it is easy to demonstrate that by formal notation we have

$$\text{MNT} = (\text{GGPT} - \text{SMNT}). \tag{8}$$

We must emphasize that in this expression the subtraction must be performed successively "row-by-row." This process can be represented by the following diagram:

$$\text{Row } k \text{ of MNT} \xrightarrow{\text{rule (6)}} \text{Row } k+2 \text{ of SMNT}$$
$$\xrightarrow{\text{rule (7)}} \text{Row } k+1 \text{ of GPT} \xrightarrow{\text{rule (8)}}$$
$$\longrightarrow \text{Row k + 1 of GGPT} - \text{Row k + 1 of SMNT} = \text{Row k + 1 of MNT}.$$

An illustration of this procedure is shown in the following construction:

k	GGPT						
0	1						
1	1	1					
2	1	3	1				
3	1	7	5	1			
4	1	13	16	7	1		
5	1	21	40	29	9	1	
6	1	31	85	91	46	11	1

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k	SMNT						
0	0						
1	0	0					
2	0	0	0				
3	0	1	0	0			
4	0	3	1	0	0		
5	0	6	5	1	0	0	
6	0	10	15	7	1	0	0

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k	MNT						
0	1						
1	1	1					
2	1	3	1				
3	1	6	5	1			
4	1	10	15	7	1		
5	1	15	35	28	9	1	
6	1	21	70	84	45	11	1

Studying the above construction, it is easy to prove that the following recurrence equation:

$$p_{k,m} = p_{k-1,m-1} + 2p_{k-1,m} - p_{k-2,m} \tag{9}$$

holds for the coefficients of the MNT with $k = 0, 1, 2, \ldots, n$ and $m = 0, 1, \ldots, k$, where $p_{r,s} = 0$ if $r < 0$ and/or $s > 0$. For example, isf we fix $k = 5$ and $m = 3$, then we obtain

$$p_{5,3} = p_{4,2} + 2p_{4,3} - p_{3,3} = 28.$$

The above construction leads to important simplifications in determinating the transfer functions of a ladder network with a large number of interacting cells. Some other interesting results may be obtained by considering special diagonals of the usual Pascal Triangle or a particular direct formula for successive rows of the MNT. The work in this direction is under development, and further results will be published soon.

4. NUMBER OF TERMS IN THE TRANSFER FUNCTION AND THE FIBONACCI SEQUENCE

To each ladder network can be attributed a corresponding signal flow graph by virtue of which the transfer function from the source node to a sink node can be determined. In the signal flow graph of a ladder network, there are no loops and, consequently, the total transfer function simplifies to the form of expression (3). On the other hand, the signal flow graph of a ladder network can be represented by an oriented graph attributed to the MNT (see Fig, 2).

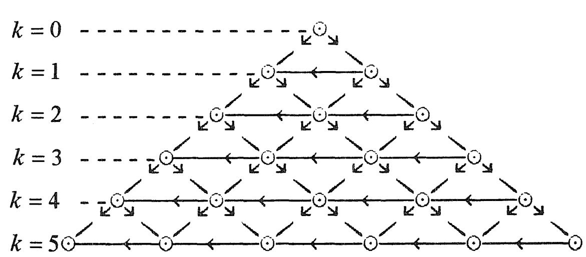

Figure 2

Each oriented branch of this graph is labeled by a transmittance equal to one. The resulting node coefficients correspond to the respective coefficients of the MNT and, simultaneously, to the number of open paths in the signal flow graph counted from the source node (the top of the graph) to the sink node (the given node in the graph). The presented graph is very useful for determining all paths appearing in the total transfer function of a ladder network containing a large number of interacting cells as, for instance, a dozen or several dozen. Moreover, it gives a possibility to answer the following question, among others: How many different open paths and corresponding total transfer functions appear in the signal flow graph and in the ladder network, respectively? It must be noted that, in the case of a quite simple ladder network, the number of open paths in the total transmittance increases rapidly with the number of cells. It can be determined by suitable use of the Fibonacci sequence that

$$F_{k+1} = F_k + F_{k-1}, \; k = 0, 1, 2, \dots, n, \tag{10}$$

with $F_0 = 0$ and $F_1 = 1$.

From Binet's formula, we have

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right], \; k - 0, 1, 2, \dots, n. \tag{11}$$

The form of this expression can be simplified by taking into consideration the Newton expansion of a power of a binomial in which the values $a = 1$ and $b = +\sqrt{5}$ or $-\sqrt{5}$ are substituted. Finally, we obtain

$$F_k = \frac{1}{2^{k-1}}\left[\binom{k}{1} + 5\binom{k}{3} + 5^2\binom{k}{5} + \cdots 5^r\binom{k}{2r+1} + \cdots + \right]. \tag{12}$$

Note that the right-hand side of this expression vanishes for $2r+1 > k$. For instance, at $k = 7$, the first vanishing term corresponds to $2r+1 > 7$, i.e., $r \geq 4$. In this case, the value of F_k amounts to $F_7 = 21$ and is composed of four terms. Moreover, a direct inspection of the oriented graph shown in Figure 2 points to a relationship between the number of open paths appearing in the total transmittance of a given ladder network. It is equal to the sum of all paths counted from the top node to all sink nodes attributed to a given level in the oriented graph representing the MNT. For a ladder network composed of n cells, the total number of terms which determine the transfer function $T_r(x)$ at the input of the r^{th} cell is equal to F_k given by (12). If the voltage to voltage ratio is computed, then we must take $k = 2r+1$ and when the current to current ratio is determined, then $k = 2r$. For example, if a network consists of eight cells, then the total number of terms in the voltage transfer function $T_{v8}(x)$ is equal to

$$S_{v8} = F_{17} = \frac{1}{2^{16}}\left[\binom{17}{1} + 5\binom{17}{3} + 5^2\binom{17}{5} + \cdots + 5^8\binom{17}{17}\right] = 1597.$$

This number determines, simultaneously, the sum of all coefficients in the MNT at the level $k = 8$. The result can be easily checked by direct inspection of the MNT up to the 8^{th} level.

5. CONCLUSIONS

The Fibonacci sequence (10) has been effectively applied to the analysis of ladder networks consisting of identical interacting cells. It has been shown that the modified numerical triangle corresponds to the respective polynomials determining the transfer functions in the network. Mapping the MNT by an oriented graph gives a possibility to evaluate all coefficients in the transfer function and the total number of terms appearing in this function. This number is expressed by the Fibonacci sequence F_{2k+1} for the voltage transfer function and by the Fibonacci sequence F_{2k} if the current transfer function is determined.

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