A GENERALIZATION OF MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

Recently Ferri, Faccio, & D'Amico ([1], [2]) introduced and studied two numerical triangles, named the DFF and the DFFz triangles. In this note, we shall see that the polynomials generated by the rows of these triangles (see [1] and [2]) are the Morgan-Voyce polynomials, which are well known in the study of electrical networks (see [3], [4], [5], and [6]). We begin this note by a generalization of these polynomials.

2. THE GENERALIZED MORGAN-VOYCE POLYNOMIALS

Let us define a sequence of polynomials $\{P_n^{(r)}\}$ by the recurrence relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x),\ n \geq 2, \tag{1}$$

with $P_0^{(r)}(x) = 1$ and $P_1^{(r)}(x) = x + r + 1$.

Here and in the sequel, r is a fixed real number. It is clear that

$$P_n^{(0)} = b_n \tag{2}$$

and that

$$P_n^{(1)} = B_n, \tag{3}$$

where b_n and B_n are the classical Morgan-Voyce polynomials (see [3], [4], [5], and [6]). We see by induction that there exists a sequence $\{a_{n,k}^{(r)}\}_{n\geq 0,\, k\geq 0}$ of numbers such that

$$P_n^{(r)}(x) = \sum_{k\geq 0} a_{n,k}^{(r)} x^k,$$

with $a_{n,k}^{(r)} = 0$ if $k > n$ and $a_{n,n}^{(r)} = 1$ if $n \geq 0$.

The sequence $a_{n,0}^{(r)} = P_n^{(r)}(0)$ satisfies the recurrence relation

$$a_{n,0}^{(r)} = 2a_{n-1,0}^{(r)} - a_{n-2,0}^{(r)},\ n \geq 2,$$

with $a_{0,0}^{(r)} = 1$ and $a_{1,0}^{(r)} = 1 + r$.

From this, we get that

$$a_{n,0}^{(r)} = 1 + nr,\ n \geq 0. \tag{4}$$

In particular, we have

$$a_{n,0}^{(0)} = 1,\ n \geq 0 \tag{5}$$

and

$$a_{n,0}^{(1)} = 1 + n,\ n \geq 0. \tag{6}$$

Following [1] and [2], one can display the sequence $\{a_{n,k}^{(r)}\}$ in a triangle:

$n \backslash k$	0	1	2	3	...
0	1				...
1	$1+r$	1			...
2	$1+2r$	$3+r$	1		...
3	$1+3r$	$6+4r$	$5+r$	1	...
...	...	...	...	...	...

Comparing the coefficient of x^k in the two members of (1), we see that, for $n \geq 2$ and $k \geq 1$,

$$a_{n,k}^{(r)} = 2a_{n-1,k}^{(r)} - a_{n-2,k}^{(r)} + a_{n-1,k-1}^{(r)}. \tag{7}$$

By this, we can easily obtain another recurring relation

$$a_{n,k}^{(r)} = a_{n-1,k}^{(r)} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1}^{(r)}, \quad n \geq 1, k \geq 1. \tag{8}$$

In fact, (8) is clear for $n \leq 2$ by direct computation. Supposing that the relation is true for $n \geq 2$, we get, by (7), that

$$\begin{aligned} a_{n+1,k}^{(r)} &= a_{n,k}^{(r)} + \left(a_{n,k}^{(r)} - a_{n-1,k}^{(r)}\right) + a_{n,k-1}^{(r)} \\ &= a_{n,k}^{(r)} + \sum_{\alpha=0}^{n-1} a_{\alpha,k-1}^{(r)} + a_{n,k-1}^{(r)} = a_{n,k}^{(r)} + \sum_{\alpha=0}^{n} a_{\alpha,k-1}^{(r)}, \end{aligned}$$

and the proof is complete by induction.

We recognize in (8) the recursive definition of the DFF and DFFz triangles. Moreover, using (5) and (6), we see that the sequence $\{a_{n,k}^{(0)}\}$ (resp. $\{a_{n,k}^{(1)}\}$) is exactly the DFF (resp. the DFFz) triangle. Thus, by (2) and (3), the generating polynomial of the rows of the DFF (resp. the DFFz) triangle is the Morgan-Voyce polynomial b_n (resp. B_n).

3. DETERMINATION OF THE $\{a_{n,k}^{(r)}\}$

In [1] and [2], the authors gave a very complicated formula for $\{a_{n,k}^{(0)}\}$ and $\{a_{n,k}^{(1)}\}$. We shall prove here a simpler formula that generalizes a known result [5] on the coefficients of Morgan-Voyce polynomials.

Theorem: For any $n \geq 0$ and $k \geq 0$, we have

$$a_{n,k}^{(r)} = \binom{n+k}{2k} + r\binom{n+k}{2k+1}, \tag{9}$$

where $\binom{a}{b} = 0$ if $b > a$.

Proof: If $k = 0$, the theorem is true by (4). Assume the theorem is true for $k-1$. We shall proceed by induction on n. Equality (9) holds for $n = 0$ and $n = 1$ by definition of the sequence

$\{a_{n,k}^{(r)}\}$. Assume that $n \geq 2$, and that (9) holds for the indices $n-2$ and $n-1$. By (7), we then have $a_{n,k}^{(r)} = 2a_{n-1,k}^{(r)} - a_{n-2,k}^{(r)} + a_{n-1,k-1}^{(r)} = X_{n,k} + rY_{n,k}$, where

$$X_{n,k} = 2\binom{n+k-1}{2k} - \binom{n+k-2}{2k} + \binom{n+k-2}{2k-2} \text{ and } Y_{n,k} = 2\binom{n+k-1}{2k+1} - \binom{n+k-2}{2k+1} + \binom{n+k-2}{2k-1}.$$

Recall that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1} = \binom{a-2}{b} + 2\binom{a-2}{b-1} + \binom{a-2}{b-2}.$$

From this, we have

$$X_{n,k} = 2\left(\binom{n+k-2}{2k} + \binom{n+k-2}{2k-1}\right) - \binom{n+k-2}{2k} + \binom{n+k-2}{2k-2}$$
$$= \binom{n+k-2}{2k} + 2\binom{n+k-2}{2k-1} + \binom{n+k-2}{2k-2} = \binom{n+k}{2k}.$$

In the same way, one can show that $Y_{n,k} = \binom{n+k}{2k+1}$; this completes the proof.

The following particular cases have been known for a long time (see [5]). If $r = 0$ (DFF triangle and Morgan-Voyce polynomial b_n), then

$$a_{n,k}^{(0)} = \binom{n+k}{2k}$$

and, if $r = 1$ (DFFz triangle and Morgan-Voyce polynomial B_n), then

$$a_{n,k}^{(1)} = \binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+k+1}{2k+1}.$$

Remark: The sequence $w_n = P_n^{(r)}(1)$ satisfies the recurrence relation $w_n = 3w_{n-1} - w_{n-2}$. On the other hand, the sequence $\{F_{2n}\}$, where F_n denotes the usual Fibonacci number, satisfies the same relation. From this, it is easily verified that

$$P_n^{(r)}(1) = F_{2n+2} + (r-1)F_{2n} = F_{2n+1} + rF_{2n}.$$

For instance, we have two known results (see [1] and [2]), $P_n^{(0)}(1) = F_{2n+1}$ and $P_n^{(1)}(1) = F_{2n+2}$. We also get a new result,

$$P_n^{(2)}(1) = F_{2n+2} + F_{2n} = L_{2n+1},$$

where L_n is the usual Lucas number.

4. MORGAN-VOYCE AND CHEBYSHEV POLYNOMIALS

Let us recall that the Chebyshev polynomials of the *second* kind, $\{U_n(w)\}$, are defined by the recurrence relation

$$U_n(\omega) = 2\omega U_{n-1}(\omega) - U_{n-2}(\omega), \tag{10}$$

with initial conditions $U_0(\omega) = 0$ and $U_1(\omega) = 1$. It is clear that the sequence $\{P_n^{(r)}(2\omega - 2)\}$ satisfies (10). Comparing the initial conditions, we obtain

$$P_n^{(r)}(2\omega-2) = U_{n+1}(\omega)+(r-1)U_n(\omega).$$

If $\omega=\cos t,\ 0<t<\pi$, it is well known that

$$U_n(\omega)=\frac{\sin(nt)}{\sin t}.$$

Thus, we have

$$P_n^{(r)}(2\omega-2)=\frac{\sin(n+1)t+(r-1)\sin nt}{\sin t}.$$

From this, we get the following formulas, where $\omega=\cos t=(x+2)/2$,

$$b_n(x)=P_n^{(0)}(x)=\frac{\cos(2n+1)t/2}{\cos t/2}, \tag{11}$$

$$B_n(x)=P_n^{(1)}(x)=\frac{\sin(n+1)t}{\sin t}. \tag{12}$$

Formulas (11) and (12) were first given by Swamy [6]. We also have a similar formula for $P_n^{(2)}(x)$, namely,

$$P_n^{(2)}(x)=\frac{\sin(2n+1)t/2}{\sin t/2}. \tag{13}$$

From (11) and (12), we see that the zeros x_k (resp. y_k) of the polynomial b_n (resp. B_n) are given by (see [6])

$$x_k=-4\sin^2\left(\frac{k\pi}{2n+2}\right), k=1,2,\ldots,n, \text{ and } y_k=-4\sin^2\left(\frac{(2k-1)\pi}{4n+2}\right), k=1,2,\ldots,n.$$

Similarly, the zeros z_k of the polynomial $P_n^{(2)}(x)$ are given by

$$z_k=-4\sin^2\left(\frac{k\pi}{2n+1}\right), k=1,2,\ldots,n.$$

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