A PERFECT CUBOID IN GAUSSIAN INTEGERS

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(*Submitted November 1992*)

1. A perfect cuboid (if such exists) has rational integral sides x, y, and z, with $xyz \neq 0$, such that the four equations

$$x^2 + y^2 = u^2, \; x^2 + z^2 = v^2, \; y^2 + z^2 = w^2, \text{ and } x^2 + y^2 + z^2 = \ell^2 \tag{1.1}$$

are satisfied for rational integers u, v, w, and ℓ. No such perfect cuboids are known, but their nonexistence has not been demonstrated. It is known that any six of the quantities x, y, z, u, v, w, and ℓ can be integral and that, in this case, an infinity of solutions exist (see [1] and [2]). We shall use the word "cuboid" in this case even when any square quantity is negative, and refer to the cuboid as nonreal, following Leech [2]. For example:

$$x = 63, \; y = 60, \; z^2 = -3344, \; u = 87, \; v = 25, \; w = 16, \text{ and } \ell = 65.$$

In this paper, a parametric solution will be determined that has two integral sides x and y (say), integral face diagonals u, v, and w, and integral internal diagonal ℓ. The third side z will, in general, be irrational or complex. However, by a suitable choice of the parameters, a perfect cuboid in Gaussian integers results that satisfies the requirement that $xyz \neq 0$.

2. From the equations above, we have that

$$2(x^2 + y^2 + z^2) = u^2 + v^2 + w^2 = 2\ell^2. \tag{2.1}$$

The equation $u^2 + v^2 + w^2 = 2\ell^2$ has the four-parameter solution

$$\begin{aligned} u &= 2(mt + mn + st - sn), \\ v &= 2ms + 2nt + n^2 + s^2 - m^2 - t^2, \\ w &= 2ms - 2nt + n^2 - s^2 + m^2 - t^2, \\ \ell &= m^2 + n^2 + s^2 + t^2. \end{aligned}$$

Substituting these values into equations (1.1) gives

$$\begin{aligned} x^2 &= (m^2 + n^2 + s^2 + t^2)^2 - (2ms - 2nt + n^2 - s^2 + m^2 - t^2)^2, \\ y^2 &= (m^2 + n^2 + s^2 + t^2)^2 - (2ms + 2nt + n^2 + s^2 - m^2 - t^2)^2, \\ z^2 &= (m^2 + n^2 + s^2 + t^2)^2 - (2(mt + mn + st - sn))^2. \end{aligned}$$

The first two equations give

$$\begin{aligned} x^2 &= 4(m^2 + n^2 + ms - nt)(s^2 + t^2 - ms + nt), \\ y^2 &= 4(n^2 + s^2 + ms + nt)(m^2 + t^2 - ms - nt). \end{aligned}$$

Let us put $m = ab, n = ac, s = -cd$, and $t = bd$, then $ms + nt = 0$ and

$$y^2 = 4(a^2c^2 + c^2d^2)(a^2b^2 + b^2d^2) = 4c^2b^2(a^2 + d^2)^2.$$

Hence, $y = 2bc(a^2 + d^2)$ and

$$x^2 = 4(a^2b^2 - 2abcd + a^2c^2)(c^2d^2 + 2abcd + b^2d^2)$$
$$= 4a^2d^2\left(b^2 - \frac{2bcd}{a} + c^2\right)\left(b^2 + \frac{2abc}{d} + c^2\right).$$

Write

$$b^2 - \frac{2bcd}{a} + c^2 = e^2 \tag{2.2}$$

and

$$b^2 + \frac{2abc}{d} + c^2 = f^2. \tag{2.3}$$

Putting $b^2 = 2bcd / a$ or $ab = 2cd$ in (2.2) and substituting in (2.3) gives $b^2 + 5c^2 = f^2$. In which case, $x = 2adcf$ and $z^2 = (a^2b^2 + c^2d^2 + a^2c^2 + b^2d^2)^2 - 4(ab(ac + bd) + cd(ac - bd))^2$. Therefore, we have the following parametric solution in which x, y, u, v, w, and d are all integral:

$$x = 2adcf,$$
$$y = 2bc(a^2 + d^2),$$
$$z^2 = ((a^2 + d^2)(b^2 + c^2))^2 - 4(ab(ac + bd) + cd(ac - bd))^2,$$

where $b^2 + 5c^2 = f^2$ and $ab = 2cd$ with $a \neq d$; otherwise, $z^2 = 0$.

We can tidy up this solution as follows: The equation $b^2 + 5c^2 = f^2$ has the solution

$$b = 5\alpha^2 - \beta^2,\quad c = 2\alpha\beta,\quad \text{and}\quad f = 5\alpha^2 + \beta^2.$$

The equation $ab = 2cd$ or $a(5\alpha^2 - \beta^2) = 4\alpha\beta d$ can be satisfied if $a = 4\alpha\beta$ and $d = 5\alpha^2 - \beta^2$. The solution can now be written as

$$\begin{aligned} x &= 16\alpha^2\beta^2(25\alpha^4 - \beta^4), \\ y &= 4\alpha\beta(5\alpha^2 - \beta^2)(25\alpha^4 + 6\alpha^2\beta^2 + \beta^4), \\ z^2 &= (25\alpha^4 + 6\alpha^2\beta^2 + \beta^4)^2(25\alpha^4 - 6\alpha^2\beta^2 + \beta^4)^2 \\ &\quad - 16\alpha^2\beta^2(5\alpha^2 - \beta^2)^2(25\alpha^4 + 14\alpha^2\beta^2 + \beta^4)^2. \end{aligned} \tag{2.4}$$

If $\alpha = 1$ and $\beta = 2$, we have

$$x = 576,\quad y = 520,\quad z^2 = 618849,$$

which is the smallest real cuboid with one irrational edge (see [2]).

If $\alpha = 1$ and $\beta = 3$, we have

$$x = 63,\quad y = 60,\quad z^2 = -3344,$$

which is the smallest cuboid (nonreal) in this category, according to Leech [2].

3. Looking at the form for z^2 in (2.4), we see that we cannot choose positive integral α and β to make

$$16\alpha^2\beta^2(5\alpha^2 - \beta^2)^2(25\alpha^4 + 14\alpha^2\beta^2 + \beta^4)^2 \tag{3.1}$$

zero. But we can put $25\alpha^4 - 6\alpha^2\beta^2 + \beta^4 = 0$ (say) to give

$$\frac{\alpha^2}{\beta^2} = \frac{3 \pm 4i}{25}.$$

Putting $\alpha^2 = 3 \pm 4i$ and $\beta^2 = 25$, we get $\alpha = 2 \pm i$ and $\beta = 5$. This gives, after cancelling common real factors

$$x = 96 \pm 28i = 4(24 \pm 7i),$$
$$y = 72 \pm 21i = 3(24 \pm 7i),$$
$$z = 35 \mp 120i = 5(7 \mp 24i),$$

and we have

$$x = 4, \qquad y = 3, \qquad z = \mp 5i,$$
$$x^2 + y^2 = (5)^2,$$
$$x^2 + z^2 = (3i)^2,$$
$$y^2 + z^2 = (4i)^2, \text{ and}$$
$$x^2 + y^2 + z^2 = (0)^2.$$

This is clearly so for the following Pythagorean values

$$x = 2pq, \quad y = p^2 - q^2, \quad \text{and} \quad z = i(p^2 + q^2).$$

Hence, according to the original definition, since $xyz \neq 0$, we have a perfect cuboid in Gaussian integers.

It would be interesting to know it if is possible to have a solution in Gaussian integers such that $xyzuvw\ell \neq 0$.

REFERENCES

1. W. J. A. Colman. "On Certain Semi-Perfect Cuboids." *The Fibonacci Quarterly* **26.2** (1988):54-57.
2. J. Leech. "The Rational Cuboid Revisited." *Amer. Math. Monthly* **84** (1977):518-33.

AMS Classification Numbers: 11D09

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