UNIQUE MINIMAL REPRESENTATION OF INTEGERS BY NEGATIVELY SUBSCRIPTED PELL NUMBERS

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1. BACKGROUND

In this paper, we prove the following uniqueness and minimality result for Pell numbers P_{-i} (see [3]):

Theorem: The representation of **any** integer N as

$$N = \sum_{i=1}^{\infty} a_i P_{-i} \tag{1.1}$$

where

$$\begin{cases} a_i = 0, 1, 2 \\ a_i = 2 \Rightarrow a_{i+1} = 0 \end{cases} \tag{1.2}$$

is unique and minimal.

Pell numbers P_n are defined in [3] as members of the two-way infinite Pell sequence $\{P_n\}$ satisfying the recurrence

$$P_{n+1} = 2P_n + P_{n-1}, \; P_0 = 0, \; P_1 = 1. \tag{1.3}$$

To compute terms of the sequence with positive subscripts, extend $(0, 1, \dots)$ to the right using (1.3); to compute terms of the sequence with negative subscripts, extend $(\dots, 0, 1)$ to the left using

$$P_{n-1} = -2P_n + P_{n+1}. \tag{1.4}$$

Induction may be used to establish that

$$P_{-n} = (-1)^{n+1} P_n. \tag{1.5}$$

Associated with P_n are the numbers

$$q_n = P_n + P_{n-1}, \tag{1.6}$$

where $2q_n = Q_n$, the n^{th} **Pell-Lucas number**.

From (1.3) and (1.6), it easily follows that

$$q_{n+1} = 2q_n + q_{n-1}. \tag{1.7}$$

Some of the smallest P_n and q_n are:

TABLE 1. Values of P_n and q_n

$n =$	...	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	...
$P_n =$	...	169	−70	29	−12	5	−2	1	0	1	2	5	12	29	70	169	...
$q_n =$	...	...	...	...	...	...	...	...	1	1	3	7	17	41	99	239	...

While values of q_n can be readily extended through negative values of n, for our purposes we need only positive values of n. For negative subscripts, $q_{-n} = (-1)^n q_n$.

It is a straightforward exercise to establish the sums [3, Theorem 2]

$$\sum_{i=1}^{n} P_{-2i} = \frac{1 - P_{-(2n+1)}}{2} \tag{1.8}$$

and

$$\sum_{i=1}^{n} P_{-2i+1} = \frac{-P_{-2n}}{2}. \tag{1.9}$$

Our procedure in demonstrating the truth of the Theorem is to adapt and extend the technique used in [2] for positively subscripted Pell numbers.

Two important differences between the criteria (1.2) in our Theorem for P_{-n} $(n > 0)$ and those in [2] for P_n $(n > 0)$ must be noted:

(i) In [2], $\varepsilon_i = 2 \Rightarrow \varepsilon_{i-1} = 0$, while in (1.2), $a_i = 2 \Rightarrow a_{i+1} = 0$.

(ii) In [2], $\varepsilon_1 = 0$; $\varepsilon_i = 0, 1, 2$ $(i > 1)$, while in (1.2), $a_i = 0, 1, 2$ $(i \geq 1)$.

The restriction on ε_1 in (ii) arises from the fact that, for n positive, a distinction has to be made between $P_2 = 2$ and $2P_1 = 2$ (the latter being excluded). No such difficulty occurs for negatively subscripted Pell numbers since $P_{-2} = -2$, $P_{-1} = 1$.

2. THE SEQUENCES $(a_1, a_2, \ldots, a_n)$

Let us now concentrate on the sequence of length $n \geq 1$,

$$(a_1, a_2, \ldots, a_n), \tag{2.1}$$

with conditions (1.2) attached. Write S_n for the number of sequences (2.1) with (1.2). S_0 is not defined.

Omitting commas and brackets for convenience, we may enumerate several S_n thus:

TABLE 2. Sequences Counted by S_n $(n = 1, 2, 3, 4)$

S_1	0	1	2				
S_2	00	01	02	10	11	12	20
S_3	000	001	002	010	011	012	020
	100	101	102	110	111	112	120
	200	201	202				
S_4	0000	0001	0002	0010	0011	0012	0020
	0100	0101	0102	0110	0111	0112	0120
	1000	1001	1002	1010	1011	1012	1020
	1100	1101	1102	1110	1111	1112	1120
	2000	2001	2002	2010	2011	2012	2020
	0200	0201	0202	1200	1201	1202	

Perusal of this tabulation reveals the methodical extension of the structure of the sequences of S_n to those of S_{n+1}.

Some lemmas are needed for the proof of the Theorem.

Lemma 1: $S_n = q_{n+1}$.

Proof: This equality is easily checked in Table 2 for $n = 1, 2, 3, 4$ for which $q_{n+1} = 3, 7, 17, 41$, respectively.

Proceed by induction on n. Assume the lemma is true for $n = k > 4$; that is, assume that $S_k = q_{k+1}$ $(k > 4)$. Now, to generate S_{k+1} from S_k,

(i) prefix 0 and 1 separately to each of the q_{k+1} sequences, and

(ii) prefix 2 followed by 0, by (1.2), to each of the q_k sequences.

Therefore, $S_{k+1} = 2q_{k+1} + q_k = q_{k+2}$ by (1.7). Thus, the Lemma is also valid for $n = k+1$ and the Lemma is proved.

Observe that q_{n+1} here plays the role for P_{-n} $(n > 0)$ which P_{n+1} plays for P_n $(n > 0)$ in [2].

Consider now

$$N = a_1P_{-1} + a_2P_{-2} + \cdots + a_nP_{-n}, \tag{2.2}$$

where a_i satisfy (1.2), i.e., the integer N is determined by the sequence (2.1).

Lemma 2:

(i) $1 - P_{-n} \le N \le -P_{-(n+1)}$ $\quad$ (n odd)

(ii) $1 - P_{-(n+1)} \le N \le -P_{-n}$ $\quad$ (n even).

Proof: Clearly, the maximum integer N generated by $(a_1, \ldots, a_n)$ is given by

$$\begin{array}{ll} 20202\ldots2 & (n \text{ odd}) \\ 20202\ldots20 & (n \text{ even}) \end{array}$$

which are the same, whereas the minimum integer N generated by $(a_1, \ldots, a_n)$ is given by

$$\begin{array}{ll} 0202\ldots0 & (n \text{ odd}) \\ 0202\ldots02 & (n \text{ even}) \end{array}$$

which are different.

Appealing to (1.8) and (1.9), we derive (i) and (ii) immediately.

Notice that Lemma 2 can be recast as

Lemma 2a:

(i) $-P_{-n} < N \le -P_{-(n+1)}$ $\quad$ (n odd),

(ii) $-P_{-(n+1)} < N - P_{-n}$ $\quad$ (n even).

Next, we link Lemmas 1 and 2.

Lemma 3: The q_{n+1} integers are

$$\begin{cases} 1 - P_{n+1}, \ldots, 0, \ldots, P_n & (n \text{ even}) \\ 1 - P_n, \ldots, 0, \ldots, P_{n+1} & (n \text{ odd}). \end{cases}$$

Proof:

$$\begin{aligned} q_{n+1} &= P_{n+1} + P_n \quad \text{by (1.6)} \\ &= (\text{number of integers} \leq 0) + (\text{number of integers} > 0), \end{aligned}$$

the order in the addition being determined by the parity of n.

Thus, for $n = 7$ (so $q_8 = 577$), the numbers are $-168, \ldots, 408$.

See Table 3 for numerical details for $n = 1, \ldots, 6$. (Cf. the result in [2] corresponding to Lemma 3.)

Calculation yields the following information about S_n:

TABLE 3

S_n	Integers Generated by $(a_1, \ldots, a_n)$
$S_1 = q_2 = 3$	$0, 1, 2$
$S_2 = q_3 = 7$	$-4, \ldots, -1, 0, 1, 2$
$S_3 = q_4 = 17$	$-4, \ldots, -1, 0, 1, \ldots, 12$
$S_4 = q_5 = 41$	$-28, \ldots\ldots, -1, 0, 1, \ldots, 12$
$S_5 = q_6 = 99$	$-28, \ldots\ldots, -1, 0, 1, \ldots\ldots, 70$
$S_6 = q_7 = 239$	$-168, \ldots\ldots\ldots, -1, 0, 1, \ldots\ldots, 70$

Lemma 4: n is uniquely determined by $N(a_n \neq 0)$.

Proof: This follows from Lemma 2a.

Lemma 5: a_n $(\neq 0)$ is uniquely determined by N.

Proof: Consider $N - a_n P_{-n}$, a specific integer in (2.2). The result follows.

Examples:

Lemma 2a: (i) $-P_{-7}\,(=-169) < 100 \leq -P_{-8}\,(=408)$.
Therefore, $N = 100 \Rightarrow n = 7$ (Lemma 4).

(ii) $-P_{-9}\,(=-985) < -500 \leq -P_{-8}\,(=408)$
Therefore, $N = -500 \Rightarrow n = 8$ (Lemma 4).

Lemma 5: Consider $N = P_{-1} + P_{-2} + P_{-4} + 2P_{-5} = 45$.

$$\text{Therefore, } \begin{cases} N - P_{-5} = 16 & \text{i.e., } a_5 = 1, \\ N - 2P_{-5} = -13 & \text{i.e., } a_5 = 2. \end{cases}$$

Proof of the Theorem: Combining Lemmas 1, 2, 3, 4, and 5, we see that the representation (1.1) with (1.2) is unique and minimal.

Minimality occurs since a number given by $(a_1, \ldots, a_n)$ is identical with the numbers given by $(a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots)$ when we adjoin as many zeros as we wish.

The reader is referred to:

(a) [3] for an algorithm that generates minimal representations of integers by Pell numbers with negative subscripts, and

(b) [1] for similar work relating to Fibonacci numbers.

Another approach to the proof of the Theorem is to adapt the methods used in [1] for Fibonacci numbers. Basically, this alternative treatment assumes that there are two permissible representations of N as a sum, and then demonstrates that this assumption leads to contradictions. To conserve space, we do not develop this complicated argument here, though it has some interesting ramifications. Inevitably, there will be some overlap of material in the two approaches.

Note on Maximality

As indicated in [1] for the Fibonacci case, we likewise assert that there can be **no** maximal representation of an integer by means of P_{-n}. This conviction is easy to justify from the obvious fact that

$$\sum_{i=1}^{n} a_i P_{-i} = \sum_{i=1}^{n-1} a_i P_{-i} + a_n P_{-n},$$

where $a_n = 1$ or 2, and then from successive replacements of the last term.

For instance, with $n = 6$, i.e., $a_6 = 1$ or 2, we have (say)

$$\begin{aligned} -59 &= P_{-1} + 2P_{-3} + P_{-6} \qquad (a_6 = 1) \\ &= P_{-1} + 2P_{-3} + \overbrace{2P_{-7} + P_{-8}} \\ &= P_{-1} + 2P_{-3} + 2P_{-7} + \overbrace{2P_{-9} + P_{-10}} \text{ and so on,} \end{aligned}$$

while

$$\begin{aligned} -129 &= P_{-1} + 2P_{-3} + 2P_{-6} \qquad (a_6 = 2) \\ &= P_{-1} + 2P_{-3} + \overbrace{P_{-5} - P_{-7}} \\ &= P_{-1} + 2P_{-3} + P_{-5} - \overbrace{2P_{-8} - P_{-9}} \text{ and so on.} \end{aligned}$$

Clearly, the summations extend as far as we wish, so there is no maximal representation.

REFERENCES

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