A NOTE ON THE NEGATIVE PASCAL TRIANGLE

Jack Y. Lee
Brooklyn, NY 11209
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We arrange the rising diagonals of Pascal's triangle in vertical columns so that the column sums form a Fibonacci sequence (see [1]). Let us arrange the coefficients of the expansion of $(1-x)^n$ symmetrically to the Pascal triangle, then in the resulting triangle, the negative Pascal triangle, the sums of the columns form the negative branch of the Fibonacci sequence. This is displayed in Table 1, where the u_n's stand for Fibonacci numbers.

TABLE 1. Pascal's Array and Corresponding Fibonacci Numbers

−1																				1
	1																		1	
−8		−1																1		8
	7		1														1		7	
−21		−6		−1												1		6		21
	15		5		1										1		5		15	
−20		−10		−4		−1								1		4		10		20
	10		6		3		1						1		3		6		10	
−5		−4		−3		−2		−1				1		2		3		4		5
	1		1		1		1		1		1		1		1		1		1	
−55	34	−21	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21	34	55
	...		u_{-7}	u_{-6}	u_{-5}	u_{-4}	u_{-3}	u_{-2}	u_{-1}	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7		...	

The fact that the sum of numbers in a column (diagonal) in the positive Pascal triangle is a Fibonacci number is well known. It is clear that the same holds for the negative Pascal triangle by its construction and by the relation $u_{-n} = (-1)^{n-1}u_n$.

To see that this extension of Pascal's triangle is made in a natural way, read the sequences parallel to the main diagonal from bottom right to upper left in Table 1. The sequences in the negative triangle constitute the coefficients of the expansion of $(1+x)^{-n}$, since the negative Pascal triangle in Table 2(a) is also expressed as (b) by means of the relation

$$\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}.$$

This enables us to redefine the negative Pascal triangle as the binomial coefficients of negative exponents. Similarly, the sequences parallel to the sequence 1, 2, 3, ... consist of the coefficients of the expansion of $(1-x)^{-n}$ in the extended Pascal triangle.

The array corresponding to the general second-order recurrence $u_n = cu_{n-2} + bu_{n-1}$, where b and c are nonzero integers, is given in Table 3. In this case, the sequences parallel to the main diagonal are generated by the function $(c+bx)^n$ for any integer n, and the sequences parallel to the sequence 1, b, b^2, ... are generated by the function $c^{n-1}(1-bx)^{-n}$ for any integer n.

TABLE 2. Two Expressions of the Negative Pascal Triangle

$$
\begin{array}{ccccccc}
\binom{6}{6} & & & & & & \\
 & -\binom{5}{5} & & & & & \\
\binom{5}{4} & & \binom{4}{4} & & & & \\
 & -\binom{4}{3} & & -\binom{3}{3} & & & \\
\binom{4}{2} & & \binom{3}{2} & & \binom{2}{2} & & \\
 & -\binom{3}{1} & & -\binom{2}{1} & & -\binom{1}{1} & \\
\binom{3}{0} & & \binom{2}{0} & & \binom{1}{0} & & \binom{0}{0}
\end{array}
$$

(a)

$$
\begin{array}{ccccccc}
\binom{-1}{6} & & & & & & \\
 & \binom{-1}{5} & & & & & \\
\binom{-2}{4} & & \binom{-1}{4} & & & & \\
 & \binom{-2}{3} & & \binom{-1}{3} & & & \\
\binom{-3}{2} & & \binom{-2}{2} & & \binom{-1}{2} & & \\
 & \binom{-3}{1} & & \binom{-2}{1} & & \binom{-1}{1} & \\
\binom{-4}{0} & & \binom{-3}{0} & & \binom{-2}{0} & & \binom{-1}{0}
\end{array}
$$

(b)

TABLE 3. The General Second-Order Array

$$
\begin{array}{ccccccccccc}
b^4/c^5 & & & & & & & & & & b^4 \\
 & -b^3/c^4 & & & & & & & & b^3 & \\
3b^2/c^4 & & b^2/c^3 & & & & & & b^2 & & 3cb^2 \\
 & -2b/c^3 & & -b/c^2 & & & & b & & 2cb & \\
1/c & & 1/c^2 & & 1/c & & 1 & & c & & c^2 \\
\hdashline
u_{-5} & u_{-4} & u_{-3} & u_{-2} & u_{-1} & u_0 & u_1 & u_2 & u_3 & u_4 & u_5
\end{array}
$$

REFERENCE

1. N. N. Vorob'ev. *Fibonacci Numbers.* New York: Blaisdell, 1961, p. 13.

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