THE LIMIT OF THE GOLDEN NUMBERS IS 3/2

Gregory A. Moore
West Shore Community College, Scottville, MI 49454-0277
gamoore@aol.com
(*Submitted September 1992*)

1. INTRODUCTION

The "Golden polynomials" $\{G_n(x)\}$ (defined in [2]) are Fibonacci polynomials satisfying

$$G_{n+2}(x) = x \cdot G_{n+1}(x) + G_n(x) \tag{1}$$

for $n \geq 0$, where $G_0(x) = -1$ and $G_1(x) = x - 1$. The maximal real root, g_n, of the function $G_n(x)$, can be considered to an n^{th}-dimensional golden ratio.

Our concern here is the study of the sequence $\{g_n\}$ of "golden numbers." A computer analysis of this sequence of roots indicated that the odd-indexed subsequence of $\{g_n\}$ was monotonically increasing and convergent to $3/2$ from below, while the even-indexed subsequence was monotoni-cally decreasing and convergent to $3/2$ from above.

In this paper, the implications of the computer analysis are proven correct. In the process, a number of lesser computational results are also developed. For example, the derivative of $G_n'(x)$ is bounded below by the Fibonacci number F_{n+1} on the interval $[3/2, \infty)$.

2. EXISTENCE

We begin with a simple yet useful formula.

Formula 2.1: $G_n\left(\frac{3}{2}\right) = -\left(\frac{1}{2}\right)^n$.

Proof: The formula is readily verified for $n = 1$ and $n = 2$ by direct computation:

$$G_0\left(\frac{3}{2}\right) = -1 = -\left(-\frac{1}{2}\right)^0 \quad \text{and} \quad G_1\left(\frac{3}{2}\right) = \frac{3}{2} - 1 = \frac{1}{2} = -\left(-\frac{1}{2}\right)^1.$$

We proceed by induction assuming the proposition is true for all indices less than n:

$$G_n\left(\frac{3}{2}\right) = \frac{3}{2}G_{n-1}\left(\frac{3}{2}\right) + G_{n-2}\left(\frac{3}{2}\right) = \frac{3}{2}\left(-\left(-\frac{1}{2}\right)^{n-1}\right) + \left(-\left(-\frac{1}{2}\right)^{n-2}\right)$$
$$= \left(\frac{3(-1)^{n-1}}{2^n} + \frac{(-1)^{n-1}}{2^{n-2}}\right) = (-1)^n\left(\frac{3-2^2}{2^n}\right) - \left(-\frac{1}{2}\right)^n. \qquad \square$$

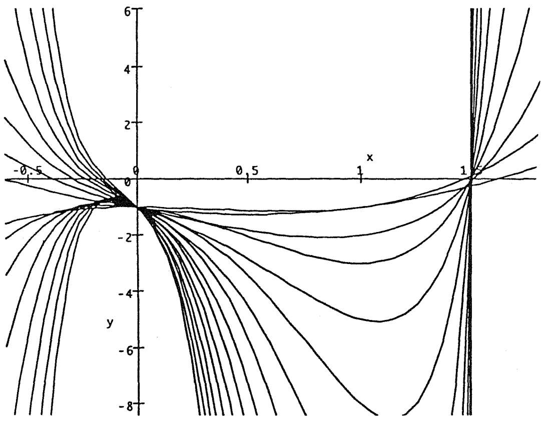
Incidentally, it is apparent from this formula that

$$\lim_{n\to\infty} G_n\left(\frac{3}{2}\right) = \lim_{n\to\infty} -\left(-\frac{1}{2}\right)^n = 0.$$

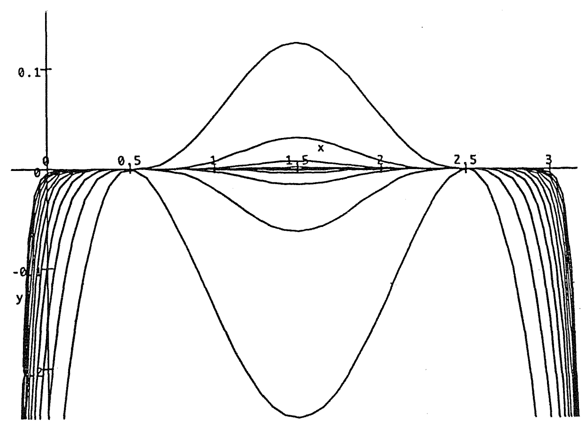
While suggestive, this is not sufficient to show the desired result about the convergence of the roots. For example, these same properties hold for the sequence of functions

$$f_n(x) = -\left(\frac{\left(x - \frac{3}{2}\right)^2 - 1}{2}\right)^n.$$

However, these roots remain at $\frac{1}{2}$ and $\frac{5}{2}$ for all values of n and do not converge to $\frac{3}{2}$.



$G_n(x)$ for $n = 2, \ldots, 17$



$f_n(x)$ for $n = 2, \ldots, 12$

Throughout this paper we will limit our discussion to polynomial functions with positive leading coefficient. These functions have the following easily proven properties.

Lemma 2.2:

A. If r is the maximal root of a function f, then $f(x) > 0$ for all $x > r$. Conversely, if $f(x) > 0$ for all $x \geq t$, then $r < t$. If $f(s) < 0$, then $s < r$.

B. Suppose R is an upper bound for the roots of the functions $f_1(x), f_2(x), \ldots, f_n(x)$, and the functions $u_0(x), u_1(x), u_2(x), \ldots, u_n(x)$ have no positive real roots. Then R is also an upper bound for the roots of the function $f(x)$ defined by

$$f(x) = f_n(x) \cdot u_n(x) + f_{n-1}(x) \cdot u_{n-1}(x) + \cdots + f_1(x) \cdot u_1(x) + u_0(x).$$

To demonstrate the existence of the sequence $\{g_n\}$, we will require two minor results from [1]. First, from Corollary 2.4, $G_n(1) = -F_{n-1}$ and $G_n(-1) = (-1)^n L_{n-1}$ [where F_{n-1} is the $(n-1)^{\text{th}}$ Fibo-nacci number and L_{n-1} is the $(n-1)^{\text{th}}$ Lucas number]. Second, from Corollary 4.3, each $G_n(x)$ is monic with constant term -1.

Proposition 2.3: Existence of $\{g_n\}$

For each $n > 1$: *A.* $G_n(x)$ has a maximum root g_n in the interval (1, 2).

B. $G_n(x)$ has no rational roots. In particular, each g_n is irrational.

Proof:

Part A. Since each G_n is monic and $G_n(1) = -F_{n-1} < 0$, then $G_n(x)$ must have a root larger than 1 (Lemma 2.2A). Since g_n is the largest root by definition, we have $g_n > 1$.

By direct computation, $G_1(x)$ and $G_2(x)$ are strictly positive on the interval $[2, \infty)$. Using the recursive relation (1) and an inductive argument, it is easy to see that each $G_n(x)$ is strictly positive on $[2, \infty)$. Therefore, $g_n < 2$ (Lemma 2.2A).

Part B. Suppose r is a rational root of $G_n(x)$, say $r = b/c$. Then G_n would be divisible by a linear factor of the form $(cx - b)$. In this case, b would divide the constant term of -1, and c would divide the leading coefficient of $+1$. The only possibilities are $\pm(x-1)$ and $\pm(x+1)$, which indicate $G_n(x)$ has a root of $+1$ or -1, respectively. However, $G_n(1) = -F_{n-1}$ and $G_n(-1) = (-1)^n L_{n-1}$. Hence, $G_n(x)$ has no rational roots. □

3. EVEN/ODD DISTINCTIONS

It is useful to note that when n is odd, $G_n(x)$ can be expressed entirely in terms of smaller odd-indexed functions and the seed function $G_0(x)$. Similarly, when n is even, we can write $G_n(x)$ in terms of smaller even-indexed functions and the seed function $G_1(x)$. Specifically, by repeated substitution, we obtain

Formula 3.1:

a. $G_{2n+1}(x) = (x^2+1)G_{2n-1}(x) + x^2 G_{2n-3}(x) + \cdots + x^2 G_1(x) + xG_0(x)$.

b. $G_{2n}(x) = (x^2+1)G_{2n-2}(x) + x^2 G_{2n-4}(x) + \cdots + x^2 G_2(x) + xG_1(x)$.

We can now show that $3/2$ is an upper bound for all of the odd-indexed g_n and a lower bound for the even-indexed g_n.

Observation 3.2: $g_{2n-1} < 3/2 < g_{2n}$ for all $n > 0$.

Proof:

Case: Even Indices. ($3/2 < g_{2n}$)

By Formula 2.1, $G_{2n}(3/2) = -2^{-2n} < 0$. Since g_n is defined to be the largest root of $G_n(x)$, the result is indicated by Lemma 2.2A.

Case: Odd Indices. ($g_{2n-1} < 3/2$)

Note that $g_1 = 1 < 3/2$. Assume then that the proposition is true for g_{2k-1} for $k < n$. Using Formula 3.1, we write

$$G_{2n+1}(x) = (x^2+1) \cdot G_{2n-1}(x) + x^2 \cdot G_{2n-3}(x) + \cdots + x^2 \cdot G_1(x) + x.$$

We can apply Lemma 2.2B because the functions x, x^2, and (x^2+1) have no positive roots, and $3/2$ is an upper bound for the roots of the $G_n(x)$ on the right side. □

4. MONOTONICITY

Formula 4.1: $G_{n+k}(g_n) = (-1)^{k+1} G_{n-k}(g_n)$.

Proof:

$\underline{k=1}$. Write (1) in the form $G_{n+1}(x) = x \cdot G_n(x) + G_{n-1}(x)$, and evaluate at $x = g_n$, noting that $G_n(g_n) = 0$.

$\underline{k=2}$. Write (1) in the forms $G_n(x) = x \cdot G_{n-1}(x) + G_{n-2}(x)$ and $G_{n+2}(x) = x \cdot G_{n+1}(x) + G_n(x)$. Now plug in $x = g_n$ and note that $G_{n-1}(g_n) = G_{n+1}(g_n)$ (the case of $k=1$) to get $G_{n-2}(g_n) = -G_{n+2}(g_n)$.

$\underline{k<j}$. Now assume the proposition is true for $k = 1, 2, \ldots, j-1$ (holding n fixed) and define A as the quotient

$$A = \frac{G_{n+k}(g_n)}{G_{n-k}(g_n)}.$$

We will show $A = (-1)^{j+1}$ to complete the proof. We can simplify A using (1) for the numerator and (1) solved for the last term, $G_n = G_{n+2} - x \cdot G_{n+1}$, for the denominator:

$$A = \frac{g_n G_{n+j-1}(g_n) + G_{n+j-2}(g_n)}{G_{n-j+2}(g_n) - g_n G_{n-j+1}(g_n)} = \frac{g_n G_{n+(j-1)}(g_n) + G_{n+(j-2)}(g_n)}{G_{n-(j-2)}(g_n) - g_n G_{n-(j-1)}(g_n)}.$$

Also define B and C and simplify using the validity of the formula for smaller values of k.

$$B = G_{n+j-1}(g_n) = G_{n-(j-1)}(g_n) = (-1)^j G_{n-(j-1)}(g_n),$$
$$C = G_{n+j-2}(g_n) = G_{n-(j-2)}(g_n) = (-1)^{j-1} G_{n-(j-2)}(g_n).$$

Substituting B and C into the simplification of A, we get

$$A = \frac{g_n B + C}{(-1)^{j-1} C - g_n (-1)^j B} = \frac{g_n B + C}{(-1)^{j+1}(C + g_n B)} = (-1)^{j+1}.$$

This shows the formula to be valid for $k = j$. □

Proposition 4.2: The subsequence of $\{g_n\}$ with odd indices is a monotonically increasing sequence; and the subsequence with even indices is monotonically decreasing.

Proof:

Odd Indices. By direct computation, $g_3 > g_1 = 1$. Assume the proposition holds up to g_{2k-1}, that is, $g_1 < g_3 < \cdots < g_{2k-3} < g_{2k-1}$. Then $G_{2k-3}(g_{2k-1}) > 0$ (Lemma 2.2A). Using Formula 4.1,

$$G_{2k+1}(g_{2k-1}) = G_{(2k-1)+2}(g_{2k-1}) = -G_{(2k-1)-2}(g_{2k-1}) = -G_{2k-3}(g_{2k-1}) < 0.$$

G_{2k+1} must have a root greater than g_{2k-1} by Lemma 2.2A. It follows that $g_{2k+1} > g_{2k-1}$.

Even Indices. Note first that $g_2 = {}^{(1+\sqrt{5})}\!/\!_2 > {}^3\!/\!_2$. Since $g_{2n-1} < {}^3\!/\!_2$ (Observation 3.2), then $G_{2n-1}(x) > 0$ on $[{}^3\!/\!_2, \infty)$. Rewriting (1), we have $G_{2n} - G_{2n-2} = x \cdot G_{2n-1} > 0$ on $[{}^3\!/\!_2, \infty)$. Thus, $G_{2n} > G_{2n-2}$ for all $x \geq {}^3\!/\!_2$; and G_{2n} has no root greater than g_{2n-2}. But $G_{2n}({}^3\!/\!_2) < 0$ by Formula

2.1. By the intermediate value theorem, there must be a root between $3/2$ and g_{2n-2}. This root must be g_{2n}. □

5. THE ODD-INDEXED CONVERGENCE

We now know that the odd-indexed $\{g_n\}$ form a monotonically increasing sequence bounded above by $3/2$, and the even-indexed $\{g_n\}$ form a monotonically decreasing sequence bounded below by $3/2$. Thus, limits do exist for both subsequences. We need two additional lemmas.

Lemma 5.1: The derivatives $G'_{2n-1}(x)$ are bounded below by F_{2n} on the interval (g_{2n-1}, ∞), where F_{2n} is the $(2n)^{\text{th}}$ Fibonacci number.

Proof: Substituting for both $G_{2n+1}(x)$ and $G_{2n-1}(x)$ in Formula 3.1, we obtain

$$\begin{aligned} G_{2n+1}(x) - G_{2n-1}(x) &= [(x^2+1)G_{2n-1}(x) + x^2G_{2n-3}(x) + \cdots + x^2G_1(x) + xG_0(x)] \\ &\quad -[(x^2+1)G_{2n-3}(x) + x^2G_{2n-5}(x) + \cdots + x^2G_1(x) + xG_0(x)] \\ &= (x^2+1)G_{2n-1}(x) - G_{2n-3}(x). \end{aligned}$$

Solving for $G_{2n+1}(x)$ gives us $G_{2n+1}(x) = (x^2+2)G_{2n-1}(x) - G_{2n-3}(x)$. Differentiating gives

$$G'_{2n+1}(x) = (x^2+2)G'_{2n-1}(x) - G'_{2n-3}(x) + 2xG_{2n-1}(x).$$

For $x > g_{2n-1}$, the last term is positive; thus, for all $x > 1$,

$$G'_{2n+1}(x) > (x^2+2)G'_{2n-1}(x) - G'_{2n-3}(x) > 3\cdot G'_{2n-1}(x) - G'_{2n-3}(x). \tag{2}$$

We compute

$$\begin{aligned} &G'_1(x) = (x-1)' = 1 = F_2, \\ &G'_3(x) = (x^3 - x^2 - 1)' = 3x^2 - 2x > 3 = F_4 \ \ (\text{for } x \ge g_3 > \sqrt{2}), \\ &G'_5(x) > 3G'_3(x) > 3(3) - 1 = 8 = F_6 \ \ (\text{for } x > g_3). \end{aligned}$$

Using induction and the Fibonacci identity $F_{2n} = 3\cdot F_{2n-2} - F_{2n-4}$, (2) becomes

$$G'_{2n+1}(x) > F_{2n+2}. \ \square$$

Actually, the growth rates of these derivatives can easily be shown to be even greater, although they are adequate for our purposes here. We are ready to demonstrate that the odd-indexed roots converge to $3/2$, with the aid of the following simple lemma.

Lemma 5.2: If polynomial functions $f(x)$ and $g(x)$ have the properties that $f(b) = g(b) > 0$ and $f'(x) > g'(x) > 0$ for all x in (a, b), then $f(x) < g(x)$ on (a, b). Furthermore, if g has a root c in (a, b), then $f(x)$ also has a root in (c, b).

Proof: Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) > 0$, which implies $h(x)$ is increasing. Since $h(x) < h(b) = 0$ for all x in (a, b), we have $f(x) - g(x) < 0$ and the first result follows. If $g(c)=0$, then $f(c)<g(c)=0$. Since $f(b)=g(b)>0$, f must have a root in the interval (c,b). □

Proposition 5.3: The odd-indexed subsequence of $\{g_n\}$ converges to $\frac{3}{2}$. That is,

$$\lim_{n\to\infty} g_{2n-1} = \frac{3}{2}.$$

Proof: Because the odd-indexed subsequence $\{g_{2n-1}\}$ is monotonically increasing and bounded above by $\frac{3}{2}$, we know that the limit exists and is less than or equal to $\frac{3}{2}$. We need only show it is no less than $\frac{3}{2}$.

We apply Lemma 5.2, setting $f(x)=G_{2n-1}(x)$ and $g(x)=x-(\frac{3}{2}-2^{2n-1})$. We note that $f(\frac{3}{2})=g(\frac{3}{2})=2^{-(2n-1)}>0$ (Formula 2.1), and $f'(x)=G'_{2n-1}(x)>F_{2n}>1=g'(x)$ (Lemma 5.1). Since $g(x)$ has a root at $x=(\frac{3}{2}-2^{2n-1})$, it follows that $G_{2n-1}(x)$ has a root in the interval $(\frac{3}{2}-2^{2n-1},\frac{3}{2})$. Thus, $\frac{3}{2}>g_{2n-1}>\frac{3}{2}-2^{2n-1}$ for all n. □

6. THE EVEN-INDEXED SUBSEQUENCE

We now address the even-indexed subsequence in a somewhat analogous way.

Lemma 6.1: The derivative $G'_{2n}(x)$ is bounded below by the Fibonacci number F_{2n+1} on $[\frac{3}{2},\infty)$.

Proof: For $x>\frac{3}{2}$, $G'_2(x)=2x-1>2(\frac{3}{2})-1=2=F_3$. Assume $G'_{2n-2}(x)>F_{2n-1}$. Differentiating (1) gives $G'_{2n}(x)=x\cdot G'_{2n-1}(x)+G'_{2n-2}(x)+G_{2n-1}(x)$.Keeping in mind that $G'_{2n-1}(x)>F_{2n}$ (Lemma 5.1) and $G_{2n-1}(\frac{3}{2})\geq 2^{-n}>0$ (Formula 2.1 and Lemma 5.1), we write

$$G'_{2n}(x)>(\tfrac{3}{2})\cdot F_{2n}+F_{2n-1}+2^{-2n-1}>F_{2n}+F_{2n-1}=F_{2n+1}. \ \square$$

Combining Lemmas 5.1 and 6.1, we have the side result

Corollary 6.2: $G'_n(x)>F_{n+1}$ on the interval $[\frac{3}{2},\infty)$.

Lemma 6.3: Suppose polynomial functions $f(x)$ and $g(x)$ have the properties $f(a)=g(a)<0$ and $f'(x)>g'(x)>0$ for all x in (a,b). Then $f(x)>g(x)$ on (a,b). Furthermore, if $g(x)$ has a root c in (a,b), then $f(x)$ also has a root in (a,c).

Proof: Apply Lemma 5.2 to the functions $-f(a+b-x)$ and $-g(a+b-x)$. □

We can now show that the even-indexed roots converge to $\frac{3}{2}$ from above.

Proposition 6.4: The even-indexed subsequence of $\{g_n\}$ converges to $\frac{3}{2}$. That is,

$$\lim_{n\to\infty} g_{2n} = \frac{3}{2}.$$

Proof: Because the sequence $\{g_{2n}\}$ is monotonically decreasing and bounded below by $\frac{3}{2}$, we know that the limit exists and is no less than $\frac{3}{2}$. We need only show that the limit is no more than $\frac{3}{2}$. Apply Lemma 6.3, letting $f(x) = G_{2n}(x)$ and $g(x) = x - (\frac{3}{2} + 2^{-2n})$. Then $f(\frac{3}{2}) = g(\frac{3}{2}) = -2^{-2n}$ and $f'(x) > F_{2n+1} > 1 = g(x)$ (Lemma 6.1). Thus, $f(x) = G_{2n}(x)$ has a root interval $(\frac{3}{2}, \frac{3}{2} + 2^{-2n})$. This means that $\frac{3}{2} < g_{2n} < \frac{3}{2} + 2^{2n-1}$ for all n. □

7. CONCLUDING REMARKS

While the golden numbers form an irrational sequence converging to $\frac{3}{2}$ with odd and even subsequences converging monotonically from below and above, respectively, there are other questions to consider. For example, computer analysis also yields the apparent approximation,

$$g_n \approx \frac{3}{2} + \Delta \cdot (-1)^{-n},$$

which could be explored. Also, it is quite likely that these results can be extended to other Fibonacci polynomial sequences. Many of the formulas and lemmas here relied only on the basic Fibonacci relationship (1) and not the specific definition of the particular functions $\{G_n\}$. Possibly there is a number like $\frac{3}{2}$ for each Fibonacci polynomial sequence.

REFERENCES

1. V. E. Hoggatt, Jr. & M. Bicknell. "Roots of Fibonacci Polynomials." *The Fibonacci Quarterly* **11.3** (1973):271-74.
2. G. Moore. "A Fibonacci Polynomial Sequence Defined by Multidimensional Continued Fractions; and Higher-Order Golden Ratios." *The Fibonacci Quarterly* **31.4** (1993):354-64.

AMS Classification Numbers: 11B39, 11B37, 11C08

❖❖❖