INFINITE PRODUCTS AND FIBONACCI NUMBERS

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In this paper we wish to describe how certain identities for infinite products lead to some striking infinite products involving terms of binary recurrences.

1. INFINITE PRODUCTS

We begin with the result on infinite products.

Theorem 1: If $|x|<1$, S is a set of positive integers and h and g are functions such that $|g(x)|$, $|h(x)|< Cx^{\alpha}$ for all x, where $C>0$ and $\alpha\geq 0$ are constants, then

$$\prod_{k\in S}(1+x^k)^{g(k)/k}(1-x^k)^{h(k)/k}=\exp\left\{-\sum_{n=1}^{+\infty}\sum_{\substack{d|n\\ d\in S}}\left(h(d)+(-1)^{n/d}g(d)\right)\frac{x^n}{n}\right\}.$$

Proof: Let

$$F(x)=\prod_{k\in S}(1+x^k)^{g(k)/k}(1-x^k)^{h(k)/k}.$$

Note that the infinite product converges absolutely for $|x|<1$. Then

$$\begin{aligned}\log F(x)&=\sum_{k\in S}\left\{\frac{g(k)}{k}\log(1+x^k)+\frac{h(k)}{k}\log(1-x^k)\right\}\\ &=\sum_{k\in S}\frac{g(k)}{k}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}x^{nk}}{n}-\sum_{k\in S}\frac{h(k)}{k}\sum_{n=1}^{+\infty}\frac{x^{nk}}{n}.\end{aligned}$$

Since $|x|<1$ and $g(k)$ and $h(k)$ are bounded by powers of k, we see that the two double series converge absolutely, and so we may interchange the order of summation. We obtain

$$\begin{aligned}\log F(x)&=-\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d|n\\ d\in S}}(-1)^{n/d}g(d)-\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d|n\\ d\in S}}h(d)\\ &=-\sum_{n=1}^{+\infty}\frac{x^n}{n}\sum_{\substack{d|n\\ d\in S}}\left(h(d)+(-1)^{n/d}g(d)\right).\end{aligned}$$

If we exponentiate, the result follows.

The following two corollaries are the results we will be using in what follows. In the first corollary, we take S to be the set of odd integers and $g=-h=f$, where f is any function that satisfies the order of magnitude bound on Theorem 1. In the second corollary, we take S to be the set of natural numbers and $g=-h=f$ as before.

Corollary 1.1: Under the hypotheses of Theorem 1, we have

$$\sum_{k=0}^{+\infty}\left(\frac{1+x^{2k+1}}{1-x^{2k+1}}\right)^{f(2k+1)/(2k+1)} = \exp\left\{2\sum_{k=0}^{+\infty}\left(\sum_{d|2k+1} f(d)\right)\frac{x^{2k+1}}{2k+1}\right\}.$$

Corollary 1.2: Under the hypotheses of Theorem 1, we have

$$\sum_{k=1}^{+\infty}\left(\frac{1+x^{k}}{1-x^{k}}\right)^{f(k)/k} = \exp\left\{\sum_{n=1}^{+\infty}\left(\sum_{d|n} f(d)\left(1-(-1)^{n/d}\right)\right)\frac{x^{n}}{n}\right\}.$$

2. BINARY RECURSIONS

Consider the binary recursion relation

$$u_{n+2} = au_{n+1} + bu_n, \quad n \geq 0, \tag{1}$$

where u_0 and u_1 are some given values. Let α and β be the roots of $x^2 - ax - b = 0$, where we take

$$\alpha = \frac{a+\sqrt{a^2+4b}}{2} \quad \text{and} \quad \beta = \frac{a-\sqrt{a^2+4b}}{2}.$$

If we assume $a > 0$ and $a^2 + 4b > 0$, then we have that

$$|\beta / \alpha| < 1. \tag{2}$$

Let $\{P_n\}$ be the solution to the recursion (1) with initial conditions $P_0 = 0$ and $P_1 = 1$. Then it is well known that we may write

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{3}$$

If we let $\{Q_n\}$ be the solution to (1) with $Q_0 = 2$ and $Q_1 = a$, then we have

$$Q_n = \alpha^n + \beta^n. \tag{4}$$

The most well known of these sequences are the Fibonacci and Lucas numbers that satisfy (1) with $a = b = 1$. In this case,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

3. SOME ARITHMETIC FUNCTIONS

In our applications of Corollaries 1.1 and 1.2, we will take f to be some well-known arithmetic functions, namely, the Euler function, φ, and the Möbius function, μ. The reason for discussing these two function is that they have the following well-known properties:

$$\sum_{d|n} \varphi(d) = n \tag{5}$$

and

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n=1, \\ 0 & n>1. \end{cases} \tag{6}$$

These two results allow us to easily sum the infinite series that appear on the right-hand sides of Corollaries 1.1 and 1.2. Unfortunately, not many other arithmetic functions have such simple sums as in (5) and (6).

A generalization of the Euler function, namely, the Jordan functions, J_k, satisfies

$$\sum_{d|n} J_k(d) = n^k,$$

but this leads us to sums of the form

$$\sum_{n=1}^{+\infty} n^{k-1} x^n,$$

which have closed-form expressions of the form

$$\frac{P_k(x)}{(1-x)^k},$$

where P_k is a polynomial. For general k, the polynomial P_k is not that tractable, and so we have chosen to go with just the Euler function.

A function that generalizes both the Euler function and the Möbius function is the Ramanujan sum, $c_n(m)$, which can be defined by

$$c_n(m) = \sum_{d|(n,m)} d\mu(u/d).$$

Then we have $c_n(1) = \mu(n)$ and $c_n(0) = \varphi(n)$. The Ramanujan sum has the nice property that

$$\sum_{d|n} c_d(m) = \begin{cases} n & n|m, \\ 0 & \text{otherwise.} \end{cases}$$

If we use this in the corollaries, we end up with sums of the form

$$\sum_{d|m} x^d,$$

which are easy to deal with for individual m, but not in general.

Therefore, in what follows, we shall restrict ourselves to the use of only the Euler and Möbius functions.

4. APPLICATION OF COROLLARY 1.1

If we let $f = \varphi$ or μ, then, since $\varphi(n) \le n$ and $|\mu(n)| \le 1$, we see that we can use either of these choices in Corollary 1.1. If $|x| < 1$, then we have, by (5),

$$\prod_{k=0}^{+\infty} \left(\frac{1+x^{2k+1}}{1-x^{2k+1}} \right)^{\varphi(2k+1)/(2k+1)} = \exp\left\{ 2\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} \sum_{d|2n+1} \varphi(d) \right\} = \exp\left\{ 2\sum_{n=0}^{+\infty} x^{2n+1} \right\} = \exp\left(\frac{2x}{1-x^2} \right). \tag{7}$$

Similarly, if we use (6), we obtain, for $|x|<1$,

$$\prod_{k=0}^{+\infty}\left(\frac{1+x^{2k+1}}{1-x^{2k+1}}\right)^{\mu(2k+1)/(2k+1)}=e^{2x}. \tag{8}$$

Theorem 2: We have

$$\prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{\varphi(2k+1)/(2k+1)}=\exp\left(\frac{-2b}{a\sqrt{a^2+4b}}\right) \tag{9}$$

and

$$\prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{\mu(2k+1)/(2k+1)}=\exp\left(\frac{a^2+2b-a\sqrt{a^2+4b}}{-b}\right). \tag{10}$$

Proof: Let $x=\beta/\alpha$. By (2), we see that $|x|<1$, and so we can use (7) and (8). We have

$$\prod_{k=0}^{+\infty}\left(\frac{1+(\beta/\alpha)^{2k+1}}{1-(\beta/\alpha)^{2k+1}}\right)^{f(2k+1)/(2k+1)}=\prod_{k=0}^{+\infty}\left(\frac{\alpha^{2k+1}+\beta^{2k+1}}{\alpha^{2k+1}-\beta^{2k+1}}\right)^{f(2k+1)/(2k+1)}$$
$$=\prod_{k=0}^{+\infty}\left(\frac{Q_{2k+1}}{(\alpha-\beta)P_{2k+1}}\right)^{f(2k+1)/(2k+1)}.$$

Taking $f=\varphi$ and μ gives the left-hand sides of (9) and (10), respectively.

If we put $x=\beta/\alpha$ into the right-hand side of (7), we obtain

$$\frac{2(\beta/\alpha)}{1-(\beta/\alpha)^2}=\frac{2\alpha\beta}{(\alpha^2-\beta^2)}=\frac{2(-b)}{(\alpha-\beta)P_2}=\frac{-2b/a}{\alpha-\beta}=\frac{-2b}{a\sqrt{a^2+4b}},$$

which completes the proof of (9).

To prove (10), we put $x=\beta/\alpha$ into the right-hand side of (8) and obtain

$$2\left(\frac{\beta}{\alpha}\right)=\frac{a^2+2b-a\sqrt{a^2+4b}}{-b},$$

which proves (10) and completes the proof of Theorem 2.

If we take $a=b=1$ to obtain the Fibonacci and Lucas sequences, we get the following corollary.

Corollary 2.1: We have

$$\prod_{k=0}^{+\infty}\left(\frac{L_{2k+1}}{\sqrt{5}F_{2k+1}}\right)^{\varphi(2k+1)/(2k+1)}=e^{-2\sqrt{5}}$$

and

$$\prod_{k=0}^{+\infty}\left(\frac{L_{2k+1}}{\sqrt{5}F_{2k+1}}\right)^{\mu(2k+1)/(2k+1)}=e^{-3+\sqrt{5}}.$$

5. AN IDENTITY FOR MULTIPLICATIVE FUNCTIONS

Theorem 3: Let f be a multiplicative function.

1) If n is odd, then

$$\sum_{d|n}(-1)^{n/d}f(d) = -\sum_{d|n}f(d).$$

2) If n is even, $n=2^s m,\ s\geq 1$, and m is odd, then

$$\sum_{d|n}(-1)^{n/d}f(d) = \sum_{d|n}f(d) - 2f(2^s)\sum_{s|m}f(s).$$

Proof: If n is odd and $d|n$, then n/d is also odd. Thus, if n is odd, we have

$$\sum_{d|n}(-1)^{n/d}f(d) = \sum_{d|n}(-1)f(d) = -\sum_{d|n}f(d),$$

which proves 1).

Suppose n is even and write $n=2^s m$, where $s\geq 1$ and m is an odd integer. Then

$$\sum_{d|n}(-1)^{n/d}f(d) = \sum_{\substack{d|n\\ n/d\text{ even}}} f(d) - \sum_{\substack{d|n\\ n/d\text{ odd}}}f(d) = \sum_{d|n}f(d) - 2\sum_{\substack{d|n\\ n/d\text{ odd}}}f(d).$$

Now if $d|n$ and n/d is odd, we can write $d=2^s\delta$, where $\delta|m$. Thus,

$$\sum_{d|n}(-1)^{n/d}f(d) = \sum_{d|n}f(d) - 2\sum_{\delta|m}f(2^s\delta).$$

Since f is multiplicative, we can write $f(2^s\delta)=f(2^s)f(\delta)$ and this gives 2) and completes the proof of the theorem.

The following corollary is just a rewriting of Theorem 3 in a form applicable to Corollary 1.2.

Corollary 3.1: Let f be a multiplicative function. Then, with the notation of Theorem 3, we have

$$\sum_{d|n}f(d)\left(1-(-1)^{n/d}\right) = \begin{cases} 2\sum_{d|n}f(d) & \text{if } n \text{ is odd,}\\ 2f(2^s)\sum_{d|m}f(d) & \text{if } n=2^s m \text{ is even.}\end{cases}$$

We now apply the corollary to our specific choices of function, namely, $\varphi(n)$ and $\mu(n)$. Since both of these are multiplicative, we can apply Corollary 3.1 to obtain the following result.

Corollary 3.2: We have

$$\sum_{d|n}\varphi(d)\left(1-(-1)^{n/d}\right) = \begin{cases} 2n & \text{if } n \text{ is odd,}\\ n & \text{if } n \text{ is even,}\end{cases}$$

and

$$\sum_{d|n}\mu(d)\left(1-(-1)^{n/d}\right) = \begin{cases} 2 & \text{if } n=1,\\ -2 & \text{if } n=2,\\ 0 & \text{if } n>2.\end{cases}$$

Proof: If n is odd, then we have

$$2\sum_{d|n}\varphi(d)=2n,$$

and if $n=2^s m$ is even, with $s\geq 1$ and m odd, then

$$2\varphi(2^s)\sum_{\delta|m}\varphi(\delta)=2\cdot 2^{s-1}\cdot m=2^s m=n.$$

This proves (11).

If n is odd, then we have

$$2\sum_{d|n}\mu(d)=\begin{cases}2\cdot 1=2 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases}$$

If $n=2^s m$ is even, then

$$2\mu(2^s)\sum_{\delta|m}\mu(\delta)=\begin{cases}2\mu(2^s) & \text{if } m=1,\\ 0 & \text{if } m>1,\end{cases}$$

and

$$\mu(2^s)=\begin{cases}-1 & \text{if } s=1,\\ 0 & \text{if } s>1.\end{cases}$$

If we combine these last two results, we see that

$$2\mu(2^s)\sum_{\delta|m}\mu(\delta)=\begin{cases}-2 & \text{if } s=1,\ m=1,\\ 0 & \text{otherwise.}\end{cases}$$

This proves (12) and completes the proof of the corollary.

6. APPLICATION OF COROLLARY 1.2

If we proceed as we did in section 4 and now apply Corollary 3.2, we obtain the following theorem and corollary.

Theorem 4: We have

$$\prod_{k=1}^{+\infty}\left(\frac{Q_k}{(\alpha-\beta)P_k}\right)^{\varphi(k)/k}=\exp\left(\frac{\beta^2+2\alpha\beta}{\alpha^2-\beta^2}\right) \quad\text{and}\quad \prod_{k=1}^{+\infty}\left(\frac{Q_k}{(\alpha-\beta)P_k}\right)^{\mu(k)/k}=\exp\left(\frac{2\alpha\beta-\beta^2}{\alpha^2}\right).$$

Corollary 4.1: We have

$$\prod_{k=1}^{+\infty}\left(\frac{L_k}{\sqrt{5}F_k}\right)^{\varphi(k)/k}=e^{-(1+\sqrt{5})/2\sqrt{5}} \quad\text{and}\quad \prod_{k=1}^{+\infty}\left(\frac{L_k}{\sqrt{5}F_k}\right)^{\mu(k)/k}=e^{(-13+5\sqrt{5})/2}.$$

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