

ON THE GREATEST INTEGER FUNCTION AND LUCAS SEQUENCES

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In 1972, Anaya & Crump [1] proved, for the Fibonacci numbers F_n , that

$$\left[\alpha^k F_n + \frac{1}{2} \right] = F_{n+k}, \quad n \geq k > 1, \quad (1)$$

where $\alpha = (1 + \sqrt{5})/2$ and $[x]$ denotes the greatest integer $\leq x$. Carlitz [2] later proved, for the sequence of Lucas numbers L_n , that

$$\left[\alpha^k L_n + \frac{1}{2} \right] = L_{n+k}, \quad n \geq k + 2, \quad k \geq 2. \quad (2)$$

Let P and Q be relatively prime integers with $P > 0$ and $D = P^2 - 4Q > 0$. Let α and β , $\alpha > \beta$, be the roots of $x^2 - Px + Q = 0$; the Lucas sequences are defined, for $n \geq 0$, by

$$U_n = U_n(P, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = V_n(P, Q) = \alpha^n + \beta^n.$$

In 1975, Everett [3] showed that, if $Q = -1$, then

$$\left[\alpha^k U_n + \frac{P}{P+1} \right] = U_{n+k} \text{ or } U_{n+k} + 1, \quad n \geq k \geq 2,$$

with the latter value obtaining when n and k are odd and $1/(P+1) \leq |\beta|^n U_k$.

The results of (1) and (2) can be extended to all Lucas sequences $\{U_n\}$ and $\{V_n\}$ with $Q = \pm 1$, and, interestingly, in view of Everett's result, with no restrictions on n or k for $n \geq k \geq 2$. It seems, also, not to have been recognized, even for the case where $P = 1, Q = -1$ (i.e., for the sequences of Fibonacci and Lucas numbers), that the existence of the relations for a given pair, P, Q , for the sequence $\{V_n\}$ implies the existence of the corresponding relations for the sequence $\{U_n\}$. We show this dependence and obtain the extension of (1) and (2) to all Lucas sequences with $Q = \pm 1$ and $n \geq k \geq 1$.

The proofs are straightforward. We recall that $[b] = a$ iff $0 \leq b - a < 1$.

Lemma: Let k and n be integers, where $n \geq k \geq 1$, and let t be a real number, $0 \leq t < \sqrt{D}/2$. If $[\alpha^k V_n + t] = V_{n+k}$, then $[\alpha^k U_n + 1/2] = U_{n+k}$.

Proof: Let $A = \alpha^k V_n - V_{n+k}$ and assume $[\alpha^k V_n + t] = V_{n+k}$. Then $0 \leq \alpha^k V_n + t - V_{n+k} < 1$; that is, $-t \leq A < 1 - t$. Now,

$$A = \alpha^k V_n - V_{n+k} = \alpha^k (\alpha^n + \beta^n) - (\alpha^{n+k} + \beta^{n+k}) = \beta^n (\alpha^k - \beta^k)$$

and

$$\alpha^k U_n - U_{n+k} = \alpha^k (\alpha^n - \beta^n) / \sqrt{D} - (\alpha^{n+k} - \beta^{n+k}) / \sqrt{D} = \beta^n (\beta^k - \alpha^k) / \sqrt{D}.$$

Thus, $\alpha^k U_n - U_{n+k} = -A/\sqrt{D}$, and $-t \leq A < 1-t$ implies

$$(t-1)/\sqrt{D} < -A/\sqrt{D} \leq t/\sqrt{D}. \tag{3}$$

Noting that $D = P^2 \pm 4 \geq 5$, it follows from (3) that, if $0 \leq t < \sqrt{D}/2$, then $-1/2 < -A/\sqrt{D} < 1/2$; hence, $0 < \alpha^k U_n + 1/2 - U_{n+k} < 1$, establishing the Lemma.

In the following theorem, values of t are given such that $[\alpha^k V_k + t] = V_{n+k}$ for all $n \geq k \geq 1$. With one exception, $(P, Q, k, n) = (1, -1, 1, 1)$, we have $0 \leq t < \sqrt{D}/2$; we observe, in particular, in (f), $7/5 < \sqrt{8}/2 \leq \sqrt{D}/2$ for $Q = -1$ and $P \geq 2$, and in (g), $1.1 < \sqrt{5}/2 = \sqrt{D}/2$ for $Q = -1$ and $P = 1$.

Theorem 1:

- (a) $[\alpha^k V_n + 1/2] = V_{n+k}$ if $Q = \pm 1, n \geq k+2, k \geq 1$, and $(P, k, n) \neq (1, 1, 3)$;
- (b) $[\alpha^k V_n + 1/2] = V_{n+k}$ if $Q = 1, n = k+1, k \geq 1$;
- (c) $[\alpha^k V_n + 1] = V_{n+k}$ if $\begin{cases} (P, k, n) = (1, 1, 3), \text{ or} \\ Q = -1, n = k+1, n \text{ odd}, k \geq 1; \end{cases}$
- (d) $[\alpha^k V_n] = V_{n+k}$ if $Q = -1, n = k+1, n \text{ even}, k \geq 1$;
- (e) $[\alpha^n V_n] = V_{2n}$ if $Q = 1$, or $Q = -1$ and n is even;
- (f) $[\alpha^n V_n + 7/5] = V_{2n}$ if $Q = -1$ and n is odd;
- (g) $[\alpha^n V_n + 1.1] = V_{2n}$ if $Q = -1, P = 1$, and n is odd, $n > 1$.

Proof: Let $Q = \pm 1$. Since $P > 0, D \geq 5$, and $1/\alpha = 2/(P + \sqrt{D})$, we have $0 < 1/\alpha \leq 2/(1 + \sqrt{5}) < .62$ for all P , and $1/\alpha < 2/(2 + \sqrt{5}) < 1/2$ if $P \geq 2$. We show that the relation $[b] = a$ holds in each case by showing that $|b - a - 1/2| < 1/2$. For any t ,

$$\left| \alpha^k V_n - V_{n+k} + t - \frac{1}{2} \right| = \left| \beta^n (\alpha^k - \beta^k) + t - \frac{1}{2} \right| = \left| Q^n (1/\alpha^{n-k} - Q^k / \alpha^{n+k}) + t - \frac{1}{2} \right|. \tag{4}$$

Case 1. $n \geq k+2, k \geq 1, t = 1/2, (P, k, n) \neq (1, 1, 3)$. By (4),

$$\left| \alpha^k V_n - V_{n+k} + t - \frac{1}{2} \right| = \left| Q^n (1/\alpha^{n-k} - Q^k / \alpha^{n+k}) \right| \leq \left| 1/\alpha^{n-k} \right| + \left| 1/\alpha^{n+k} \right|.$$

If $P \geq 2$, this sum is $< (1/2)^2 + (1/2)^3 < 1/2$, and if $P = 1$ and $n \geq 4$, the sum is $\leq (.62)^2 + (.62)^3 < 1/2$; this proves (a).

Case 2. $n = k+1, k \geq 1$. If $Q = 1$ and $t = 1/2$, (4) equals $|1/\alpha - 1/\alpha^{2n-1}|$. Since $D = P^2 - 4 > 0, P \geq 3$, implying that $0 < 1/\alpha < 1/2$; hence, $|1/\alpha - 1/\alpha^{2n-1}| = 1/\alpha - 1/\alpha^{2n-1} < 1/\alpha < 1/2$, proving (b). If $(P, k, n) = (1, 1, 3)$, then $0 < P^2 - 4Q = 1 - 4Q$ implies $Q = -1$, and

$$\alpha^k V_n + 1 = \alpha^1 L_3 + 1 = 4 \cdot (1 + \sqrt{5})/2 + 1 \approx 7.472;$$

thus, $[\alpha L_3 + 1] = 7 = V_4$. If $Q = -1, t = 1, n = k+1, k \geq 1$, and n is odd, (4) equals

$$\left| -1/\alpha + (-1)^k / \alpha^{2n-1} + \frac{1}{2} \right| = \left| 1/\alpha - (-1)^k / \alpha^{2n-1} - \frac{1}{2} \right|.$$

Since $n \geq 3$, $0 < 1/\alpha \pm 1/\alpha^{2n-1} < .62 + (.62)^5 < 1$, so $|1/\alpha - (-1)^k / \alpha^{2n-1} - 1/2| < 1/2$, proving (c). If $Q = -1, t = 0$, and n is even, (4) equals $|1/\alpha - (-1)^k / \alpha^{2n-1} - 1/2|$. Since $0 < 1/\alpha \pm 1/\alpha^{2n-1} < .62 + (.62)^3 < 1$, $|1/\alpha - (-1)^k / \alpha^{2n-1} - 1/2| < 1/2$, proving (d).

Case 3. $n = k$. In this case, (4) is $|Q^n(1 - (Q/\alpha^2)^n) + t - 1/2|$. If $Q = 1$ and $t = 0$, this equals $|1/2 - (1/\alpha^2)^n| < 1/2$, proving (e) for $Q = 1$; if $Q = -1, t = 0$, and n is even, (4) has exactly the same value as for $Q = 1, t = 0$, completing the proof of (e). If $Q = -1, t = 7/5$, and n is odd, (4) equals

$$\left| -\left(1 + \frac{1}{\alpha^{2n}}\right) + \frac{9}{10} \right| = \frac{1}{\alpha^{2n}} + \frac{1}{10} < (.62)^2 + .10 < \frac{1}{2},$$

proving (f). If $Q = -1, P = 1, t = 1.1$, and $n > 1$ is odd, then (4) equals

$$\left| -\left(1 + \frac{1}{\alpha^{2n}}\right) + \frac{11}{10} - \frac{1}{2} \right| = \left| -.40 - \frac{1}{\alpha^{2n}} \right| = \frac{1}{\alpha^{2n}} + .40 < (.62)^6 + .40 < \frac{1}{2},$$

establishing the last relation of the theorem.

As noted in the paragraph preceding Theorem 1, the hypothesis of the Lemma is satisfied for $n \geq k \geq 1$, with one exception, yielding the following theorem.

Theorem 2: If $Q = \pm 1$ and $n \geq k \geq 1$, then $[\alpha^k U_n + 1/2] = U_{n+k}$ with the single exception $U_n = F_n$ with $n = k = 1$.

It should perhaps be mentioned that the exception was properly excluded in (1) at the beginning of our paper, but that the case $n = k = 1$ was mistakenly included in [1]. In the interest of completeness, we observe that $[\alpha F_1] = [(1 + \sqrt{5})/2] = 1 = F_2$.

Example 1: Let $P = 3, Q = -1, n = 5, k = 4$. The first ten terms of $\{U_n(3, -1)\}$ ($0 \leq n \leq 9$) are 0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970. Therefore, $U_9 = 12970$. Since $\alpha^2 - P\alpha + Q = 0$, $\alpha^2 = 3\alpha + 1$, and $\alpha^4 = 9\alpha^2 + 6\alpha + 1 = 33\alpha + 10$. (It is easy to show, incidentally, that $\alpha^r = U_r\alpha - QU_{r-1}$ for $r > 0$.) Hence,

$$\alpha^4 U_5 + \frac{1}{2} = \left(33 \left(\frac{3 + \sqrt{13}}{2} \right) + 10 \right) 109 + \frac{1}{2} \approx 12970.58397,$$

showing that $[\alpha^4 U_5 + 1/2] = U_9$.

Example 2: Let $P = 6, Q = 1, n = k = 4$. Using $\alpha^2 = 6\alpha - 1$, we find that $\alpha^4 V_4 = 1331714.99^+$, implying $V_8 = 1331714$, by Theorem 1(e). This agrees with the result obtained using the well-known formula $V_{2n} = V_n^2 - 2Q^n$, recursively, for $n = 1, 2$, and 4.

REFERENCES

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