ON THE GREATEST INTEGER FUNCTION AND LUCAS SEQUENCES

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In 1972, Anaya & Crump [1] proved, for the Fibonacci numbers F_n, that

$$\left[\alpha^k F_n + \frac{1}{2}\right] = F_{n+k},\ n \ge k > 1, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$ and $[x]$ denotes the greatest integer $\le x$. Carlitz [2] later proved, for the sequence of Lucas numbers L_n, that

$$\left[\alpha^k L_n + \frac{1}{2}\right] = L_{n+k},\ n \ge k+2,\ k \ge 2. \tag{2}$$

Let P and Q be relatively prime integers with $P > 0$ and $D = P^2 - 4Q > 0$. Let α and β, $\alpha > \beta$, be the roots of $x^2 - Px + Q = 0$; the Lucas sequences are defined, for $n \ge 0$, by

$$U_n = U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = V_n(P,Q) = \alpha^n + \beta^n.$$

In 1975, Everett [3] showed that, if $Q = -1$, then

$$\left[\alpha^k U_n + \frac{P}{P+1}\right] = U_{n+k} \text{ or } U_{n+k} + 1,\ n \ge k \ge 2,$$

with the latter value obtaining when n and k are odd and $1/(P+1) \le |\beta|^n U_k$.

The results of (1) and (2) can be extended to all Lucas sequences $\{U_n\}$ and $\{V_n\}$ with $Q = \pm 1$, and, interestingly, in view of Everett's result, with no restrictions on n or k for $n \ge k \ge 2$. It seems, also, not to have been recognized, even for the case where $P = 1, Q = -1$ (i.e., for the sequences of Fibonacci and Lucas numbers), that the existence of the relations for a given pair, P, Q, for the sequence $\{V_n\}$ implies the existence of the corresponding relations for the sequence $\{U_n\}$. We show this dependence and obtain the extension of (1) and (2) to all Lucas sequences with $Q = \pm 1$ and $n \ge k \ge 1$.

The proofs are straightforward. We recall that $[b] = a$ iff $0 \le b - a < 1$.

Lemma: Let k and n be integers, where $n \ge k \ge 1$, and let t be a real number, $0 \le t < \sqrt{D}/2$. If $[\alpha^k V_n + t] = V_{n+k}$, then $[\alpha^k U_n + 1/2] = U_{n+k}$.

Proof: Let $A = \alpha^k V_n - V_{n+k}$ and assume $[\alpha^k V_n + t] = V_{n+k}$. Then $0 \le \alpha^k V_n + t - V_{n+k} < 1$; that is, $-t \le A < 1 - t$. Now,

$$A = \alpha^k V_n - V_{n+k} = \alpha^k(\alpha^n + \beta^n) - (\alpha^{n+k} + \beta^{n+k}) = \beta^n(\alpha^k - \beta^k)$$

and

$$\alpha^k U_n - U_{n+k} = \alpha^k(\alpha^n - \beta^n)/\sqrt{D} - (\alpha^{n+k} - \beta^{n+k})/\sqrt{D} = \beta^n(\beta^k - \alpha^k)/\sqrt{D}.$$

Thus, $\alpha^k U_n - U_{n+k} = -A/\sqrt{D}$, and $-t \le A < 1-t$ implies

$$(t-1)/\sqrt{D} < -A/\sqrt{D} \le t/\sqrt{D}. \tag{3}$$

Noting that $D = P^2 \pm 4 \ge 5$, it follows from (3) that, if $0 \le t < \sqrt{D}/2$, then $-1/2 < -A/\sqrt{D} < 1/2$; hence, $0 < \alpha^k U_n + 1/2 - U_{n+k} < 1$, establishing the Lemma.

In the following theorem, values of t are given such that $[\alpha^k V_k + t] = V_{n+k}$ for all $n \ge k \ge 1$. With one exception, $(P, Q, k, n) = (1, -1, 1, 1)$, we have $0 \le t < \sqrt{D}/2$; we observe, in particular, in (f), $7/5 < \sqrt{8}/2 \le \sqrt{D}/2$ for $Q = -1$ and $P \ge 2$, and in (g), $1.1 < \sqrt{5}/2 = \sqrt{D}/2$ for $Q = -1$ and $P = 1$.

Theorem 1:

(a) $\left[\alpha^k V_n + 1/2\right] = V_{n+k}$ if $Q = \pm 1, n \ge k+2, k \ge 1$, and $(P, k, n) \ne (1, 1, 3)$;

(b) $\left[\alpha^k V_n + 1/2\right] = V_{n+k}$ if $Q = 1, n = k+1, k \ge 1$;

(c) $[\alpha^k V_n + 1] = V_{n+k}$ if $\begin{cases}(P, k, n) = (1, 1, 3), \text{ or} \\ Q = -1,\ n = k+1,\ n \text{ odd},\ k \ge 1;\end{cases}$

(d) $[\alpha^k V_n] = V_{n+k}$ if $Q = -1, n = k+1, n$ even, $k \ge 1$;

(e) $[\alpha^n V_n] = V_{2n}$ if $Q = 1$, or $Q = -1$ and n is even;

(f) $\left[\alpha^n V_n + 7/5\right] = V_{2n}$ if $Q = -1$ and n is odd;

(g) $[\alpha^n V_n + 1.1] = V_{2n}$ if $Q = -1$, $P = 1$, and n is odd, $n > 1$.

Proof: Let $Q = \pm 1$. Since $P > 0, D \ge 5$, and $1/\alpha = 2/(P + \sqrt{D})$, we have $0 < 1/\alpha \le 2/(1+\sqrt{5}) < .62$ for all P, and $1/\alpha < 2/(2+\sqrt{5}) < 1/2$ if $P \ge 2$. We show that the relation $[b] = a$ holds in each case by showing that $|b - a - 1/2| < 1/2$. For any t,

$$\left|\alpha^k V_n - V_{n+k} + t - \frac{1}{2}\right| = \left|\beta^n(\alpha^k - \beta^k) + t - \frac{1}{2}\right| = \left|Q^n(1/\alpha^{n-k} - Q^k/\alpha^{n+k}) + t - \frac{1}{2}\right|. \tag{4}$$

Case 1. $n \ge k+2, k \ge 1, t = 1/2, (P, k, n) \ne (1, 1, 3)$. By (4),

$$\left|\alpha^k V_n - V_{n+k} + t - \frac{1}{2}\right| = \left|Q^n(1/\alpha^{n-k} - Q^k/\alpha^{n+k})\right| \le \left|1/\alpha^{n-k}\right| + \left|1/\alpha^{n+k}\right|.$$

If $P \ge 2$, this sum is $< (1/2)^2 + (1/2)^3 < 1/2$, and if $P = 1$ and $n \ge 4$, the sum is $\le (.62)^2 + (.62)^5 < 1/2$; this proves (a).

Case 2. $n = k+1, k \ge 1$. If $Q = 1$ and $t = 1/2$, (4) equals $|1/\alpha - 1/\alpha^{2n-1}|$. Since $D = P^2 - 4 > 0$, $P \ge 3$, implying that $0 < 1/\alpha < 1/2$; hence, $|1/\alpha - 1/\alpha^{2n-1}| = 1/\alpha - 1/\alpha^{2n-1} < 1/\alpha < 1/2$, proving (b). If $(P, k, n) = (1, 1, 3)$, then $0 < P^2 - 4Q = 1 - 4Q$ implies $Q = -1$, and

$$\alpha^k V_n + 1 = \alpha^1 L_3 + 1 = 4 \cdot (1+\sqrt{5})/2 + 1 \approx 7.472;$$

thus, $[\alpha L_3 + 1] = 7 = V_4$. If $Q = -1, t = 1, n = k+1, k \ge 1$, and n is odd, (4) equals

$$\left|-1/\alpha+(-1)^k/\alpha^{2n-1}+\frac{1}{2}\right|=\left|1/\alpha-(-1)^k/\alpha^{2n-1}-\frac{1}{2}\right|.$$

Since $n\geq 3$, $0<1/\alpha\pm 1/\alpha^{2n-1}<.62+(.62)^5<1$, so $|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|<1/2$, proving (c). If $Q=-1, t=0$, and n is even, (4) equals$|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|$. Since $0<1/\alpha\pm 1/\alpha^{2n-1}<$ $.62+(.62)^3<1$, $|1/\alpha-(-1)^k/\alpha^{2n-1}-1/2|<1/2$, proving (d).

Case 3. $n=k$. In this case, (4) is $|Q^n(1-(Q/\alpha^2)^n)+t-1/2|$. If $Q=1$ and $t=0$, this equals $|1/2-(1/\alpha^2)^n|<1/2$, proving (e) for $Q=1$; if $Q=-1$, $t=0$, and n is even, (4) has exactly the same value as for $Q=1, t=0$, completing the proof of (e). If $Q=-1$, $t=7/5$, and n is odd, (4) equals

$$\left|-\left(1+\frac{1}{\alpha^{2n}}\right)+\frac{9}{10}\right|=\frac{1}{\alpha^{2n}}+\frac{1}{10}<(.62)^2+.10<\frac{1}{2},$$

proving (f). If $Q=-1, P=1, t=1.1$, and $n>1$ is odd, then (4) equals

$$\left|-\left(1+\frac{1}{\alpha^{2n}}\right)+\frac{11}{10}-\frac{1}{2}\right|=\left|-.40-\frac{1}{\alpha^{2n}}\right|=\frac{1}{\alpha^{2n}}+.40<(.62)^6+.40<\frac{1}{2},$$

establishing the last relation of the theorem.

As noted in the paragraph preceding Theorem 1, the hypothesis of the Lemma is satisfied for $n\geq k\geq 1$, with one exception, yielding the following theorem.

Theorem 2: If $Q=\pm 1$ and $n\geq k\geq 1$, then $[\alpha^k U_n+1/2]=U_{n+k}$ with the single exception $U_n=F_n$ with $n=k=1$.

It should perhaps be mentioned that the exception was properly excluded in (1) at the beginning of our paper, but that the case $n=k=1$ was mistakenly included in [1]. In the interest of completeness, we observe that $[\alpha F_1]=[(1+\sqrt{5})/2]=1=F_2$.

Example 1: Let $P=3, Q=-1, n=5, k=4$. The first ten terms of $\{U_n(3,-1)\}$ $(0\leq n\leq 9)$ are 0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970. Therefore, $U_9=12970$. Since $\alpha^2-P\alpha+Q=0$, $\alpha^2=3\alpha+1$, and $\alpha^4=9\alpha^2+6\alpha+1=33\alpha+10$. (It is easy to show, incidentally, that $\alpha^r=U_r\alpha-QU_{r-1}$ for $r>0$.) Hence,

$$\alpha^4U_5+\frac{1}{2}=\left(33\left(\frac{3+\sqrt{13}}{2}\right)+10\right)109+\frac{1}{2}\approx 12970.58397,$$

showing that $[\alpha^4U_5+1/2]=U_9$.

Example 2: Let $P=6, Q=1, n=k=4$. Using $\alpha^2=6\alpha-1$, we find that $\alpha^4V_4=1331714.99^+$, implying $V_8=1331714$, by Theorem 1(e). This agrees with the result obtained using the well-known formula $V_{2n}=V_n^2-2Q^n$, recursively, for $n=1,2$, and 4.

REFERENCES

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