ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-493 ***Proposed by Stefano Mascella & Piero Filipponi, Rome, Italy***

Let $P_k(d)$ denote the probability that the k^{th} digit (from left) of an ℓ digit ($\ell \geq k$) Fibonacci number F_n (expressed in base 10) whose subscript is randomly chosen within a *large* interval $[n_1, n_2]$ ($n_2 \gg n_1$) is d.

That the sequence $\{F_n\}$ obeys Benford's law is a well-known fact (e.g., see [1] and [2]). In other words, it is well known that $P_1(d) = \log_{10}(1+1/d)$.

Find an expression for $P_2(d)$.

References

1. P. Filipponi. "Some Probabilistic Aspects of the Terminal Digits of Fibonacci Numbers." *The Fibonacci Quarterly* (to appear).
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981):175-77.

H-494 ***Proposed by David M. Bloom, Brooklyn College, New York, NY***

It is well known that if $P(p)$ is the Fibonacci entry point ("rank of apparition") of the odd prime $p \neq 5$, then $P(p)$ divides $p+e$ where $e = \pm 1$. In [1] it is stated without proof [Theorem 5(b)] that the integer $(p+e)/P(p)$ has the same parity as $(p-1)/2$. Give a proof.

Reference

1. D. Bloom. "On Periodicity in Generalized Fibonacci Sequences." *Amer. Math. Monthly* **72** (1965):856-61.

H-495 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let p be a prime $\neq 2, 5$, and let $Z(p)$ denote the *Fibonacci entry-point* of p (i.e., the smallest positive integer m such that $p \mid F_m$). Prove the following "Parity Theorem" for the Fibonacci entry-point:

A. If $p \equiv 11$ or $19 \pmod{20}$, then $Z(p) \equiv 2 \pmod 4$;
B. if $p \equiv 13$ or $17 \pmod{20}$, then $Z(p)$ is odd;
C. if $p \equiv 3$ or $7 \pmod{20}$, then $4 \mid Z(p)$.

SOLUTIONS

Irrational Behavior

H-481 *Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 4, November 1993)*

Let $\phi(x)$ be the function defined by

$$\phi(x) = \sum_{n\geq 0} \frac{x^n}{F_{r^n}}$$

where $r \geq 2$ is a natural integer. Show that $\phi(x)$ is an irrational number if x is a nonzero rational number.

Solution by Norbert Jensen, Kiel, Germany

Let $x \in \mathbb{Q}\setminus\{0\}$. We have to show the irrationality of $\phi(x)$.

The proof is similar to the well-known proof of the irrationality of e. Note that the series $\sum_{n=m}^{\infty} x^n / F_{r^n}$ and $\sum_{n=m}^{\infty} x^n / \alpha^{r^n}$ converge for all $m \in \mathbb{N}_0$. This can be proved by the ratio test. For the second series, the proof is obvious. Applying the test to the first series, one can use the following step (0).

Step (0): $F_{r^n} / F_{r^{n+m}} \leq 8\alpha^{r^n(1-r^m)}$ for all $n, m \in \mathbb{N}_0$.

Proof: $F_{r^n} / F_{r^{n+m}} = (\alpha^{r^n} - \beta^{r^n}) / (\alpha^{r^{n+m}} - \beta^{r^{n+m}}) \leq (\alpha^{r^n} + 1) / (\alpha^{r^{n+m}} - 1) \leq 2\alpha^{r^n} / (1/4)\alpha^{r^{n+m}} = 8\alpha^{r^n(1-r^m)}$. Q.E.D. [(0)]

Let $\rho_m = \sum_{n=m}^{\infty} x^n / F_{r^n}$ for all $m \in \mathbb{N}$.

Step (1): For an appropriate positive constant $c \in \mathbb{R}$, we have $|\rho_m| \leq c|x|^m/F_{r^m}$ for all $m \in \mathbb{N}$. c depends only on $|x|$.

Proof: From (0), we derive

$$|\rho_m| \leq \left(\sum_{n=m}^{\infty} F_{r^m}|x|^{n-m}/F_{r^n}\right)|x|^m/F_{r^m} \leq \left(8\alpha\sum_{n=m}^{\infty}|x|^{n-m}/\alpha^{r^{n-m}}\right)|x|^m/F_{r^m} = \left(8\alpha\sum_{n=0}^{\infty}|x|^n/\alpha^{r^n}\right)|x|^m/F_{r^m}.$$

Q.E.D. [(1)]

Step (2): Let $z \in \mathbb{N}$. Then $|z^{m-1}F_{r^{m-1}}\rho_m| < 1$ for all sufficiently large $m \in \mathbb{N}$.

Proof: $|z^{m-1}F_{r^{m-1}}\rho_m| \leq cz^{m-1}F_{r^{m-1}}|x|^m/F_{r^m} = c|x|(z|x|)^{m-1}F_{r^{m-1}} / F_{r^m} \leq d(z|x|)^{m-1} / \alpha^{r^{m-1}}$ by (1), (0), where d is an appropriate positive constant depending only on $|x|$. Since $\sum_{n=0}^{\infty}(z|x|)^n / \alpha^{r^n}$ converges, the last term tends to 0 as m tends to infinity. Q.E.D. [(2)]

Step (3): There is an $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0 : \rho_m \neq 0$.

Proof: **Case 1**—$x > 0$. The assertion follows because $\rho_m \geq x^m / F_{r^m} > 0$.

Case 2—$x < 0$. Let $m_0 \in \mathbb{N}$ such that $\alpha^{r^{m_0}(r-1)} > 8|x|$. Let $m, n \in \mathbb{N}$, $n \geq m \geq m_0$. Then $\alpha^{r^n(r-1)} \geq \alpha^{r^{m_0}(r-1)} > 8|x|$. Therefore, by (0): $F_{r^{n+1}} / F_{r^n} > |x|$ and $1/F_{r^n} > |x|/F_{r^{n+1}}$, whence $|x|^n/F_{r^n} > |x|^{n+1}/F_{r^{n+1}}$.

If m is even, it follows that

$$\rho_m = \sum_{k=m/2}^{\infty} \left(|x|^{2k}/F_{r^{2k}} - |x|^{2k+1}/F_{r^{2k+1}} \right) \geq |x|^m/F_{r^m} - |x|^{m+1}/F_{r^{m+1}} > 0.$$

If m is odd, an analogous argument shows that $\rho_m < 0$. Q.E.D. [(3)]

Step (4): $\phi(x)$ is irrational.

Proof: Let $p, q \in \mathbb{Z}, q > 0$, such that $x = p/q$. Suppose on the contrary that $\phi(x)$ is rational. Then there are $a, b \in \mathbb{Z}, b > 0$, such that $\phi(x) = a/b$. Thus, $b\phi(x) \in \mathbb{Z}$.

According to (2) and (3), there is an $m \in \mathbb{N}, m \geq 2$, such that $|(bq)^{m-1} F_{r^{m-1}} \rho_m| < 1$ and $\rho_m \neq 0$. Let $\sigma_m := \sum_{n=0}^{m-1} x^n / F_{r^n}$. Now

$$(bq)^{m-1} F_{r^{m-1}} \phi(x) = (bq)^{m-1} F_{r^{m-1}} (\sigma_m + \rho_m) = (bq)^{m-1} F_{r^{m-1}} \sigma_m + (bq)^{m-1} F_{r^{m-1}} \rho_m \in \mathbb{Z}$$

and

$$(bq)^{m-1} F_{r^{m-1}} \sigma_m \in \mathbb{Z},$$

since F_{r^j} divides $F_{r^{m-1}}$ for $j = 0, 1, \ldots, m-1$. But $|(bq)^{m-1} F_{r^{m-1}} \rho_m| < 1$; hence, $(bq)^{m-1} F_{r^{m-1}} \rho_m = 0$, $\rho_m = 0$, a contradiction. Q.E.D.

Also solved by P. Bruckman, H.-J. Seiffert, and the proposer.

Generalize

H-482 ***Proposed by Larry Taylor, Rego Park, NY***
(Vol. 31, no. 4, November 1993)

Let j, k, m, and n be integers. Let $A_n(m) = B_n(m-1) + 4A_n(m-1)$ and $B_n(m) = 4B_n(m-1) + 5A_n(m-1)$ with initial values $A_n(0) = F_n$, $B_n(0) = L_n$.

(A) Generalize the numbers $(2,2,2,2,2,2,2,2,2,2,2)$ to form an eleven-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(B) Generalize the numbers $(3,3,3,3,3,3,3,3,3,3)$ to form a ten-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

(C) Generalize the numbers $(1,1,1,1,1,1,1,1)$ to form an eight-term arithmetic progression of integral multiples of $A_{n+k}(m+j)$ and / or $B_{n+k}(m+j)$ with common difference $A_n(m)$.

Hint: $A_n(1) = -11(-1)^n A_{-n}(-1)$.

Reference: L. Taylor. Problem H-422. *The Fibonacci Quarterly* **28.3** (1990):285-87.

Solution by Paul S. Bruckman, Edmonds, WA

The recurrence defining the $A_n(m)$'s and $B_n(m)$'s may be put into matrix form

$$C\underline{x}_n(m-1) = \underline{x}_n(m), \tag{1}$$

where

$$C = \begin{pmatrix} 4 & 1 \\ 5 & 4 \end{pmatrix}, \tag{2}$$

$$\underline{x}_n(j) = \begin{pmatrix} A_n(j) & B_n(j) \end{pmatrix}^T. \tag{3}$$

We may invert the recurrence in the matrix form

$$\underline{x}_n(m-1)=C^{-1}\underline{x}_n(m),\quad \text{where } C^{-1}=\frac{1}{11}\begin{pmatrix}4 & -1\\ -5 & 4\end{pmatrix}.$$

This yields the relations:

$$A_n(m-1)=\frac{1}{11}(4A_n(m)-B_n(m)),\quad B_n(m-1)=\frac{1}{11}(-5A_n(m)+4B_n(m)). \tag{4}$$

By repeated application of (1) (in either direction), we obtain

$$C^m\underline{x}_n(0)=\underline{x}_n(m),\quad \text{with } \underline{x}_n(0)=(F_n \quad L_n)^T. \tag{5}$$

We may show that there exist two sequences of rationals (ρ_m) and (σ_m) (integers for $m\geq 0$), such that

$$C^m=\begin{pmatrix}\rho_m & \sigma_m\\ 5\sigma_m & \rho_m\end{pmatrix}. \tag{6}$$

We have no need to investigate further into these sequences, except to note that they are functions solely of m, and not of n. The relevant observation from (5)-(6) is the following:

$$A_n(m)=\rho_m F_n+\sigma_m L_n,\quad B_n(m)=5\sigma_m F_n+\rho_m L_n. \tag{7}$$

Now using the identities $L_n=F_n+2F_{n-1}, 5F_n=L_n+2L_{n-1}$, and making the substitutions $\rho_m+\sigma_m=r_m, 2\sigma_m=s_m$, (7) is transformed to the following:

$$A_n(m)=r_mF_n+s_mF_{n-1},\quad B_n(m)=r_mL_n+s_mL_{n-1}. \tag{8}$$

In this form, we see that $A_n(m)$ and $B_n(m)$ are generalized Fibonacci and Lucas numbers, respectively, as these were defined in part (B) of the published solution to H-422 (see reference [1]). Here, the $A_n(m), B_n(m), r_m$, and s_m replace the U_n, V_n, r, and s, respectively, as such were introduced in [1]. Note that the $A_n(m)$ and $B_n(m)$, for fixed m, satisfy the same linear recurrences as are satisfied by F_n and L_n; in the sequel, we shall tacitly use these without comment [e.g., $A_{n+2}(m)=A_{n+1}(m)+A_n(m), B_n(m)=A_{n+1(m)}+A_{n-1}(m)$, etc.]. Also, in the sequel, we will write (for brevity) $A_n\equiv A_n(m), \overline{A}_n\equiv A_n(m+1), \underline{A}_n\equiv A_n(m-1)$, with similar notation for the $B_n(m)$'s; however, in the final solution of each part, we will revert to the unabridged notation.

Solution of Part (A)

Using parts (B) and (A1) of [1], the following 7-term arithmetic progression (A.P.) is found, whose common difference (c.d.) is equal to A_n, and whose terms consist of integral multiples of A_{n+k} and/or B_{n+k}:

$$(-2A_{n-2}, A_{n-3}, 2A_{n-1}, B_n, 2A_{n+1}, A_{n+3}, 2A_{n+2}). \tag{9}$$

Our goal, if possible, is to affix four additional terms to the A.P. above (at one or both ends), such that these terms are of the form required in the statement of the problem, such that the c.d. for all 11 terms remains A_n, and such that, for some fixed m and n, all 11 terms equal 2. We require a few additional identities:

$$2A_{n+2}+A_n=\overline{A}_n. \tag{10}$$

Proof: Replacing m by $m+1$ in the original recurrence, we have:

$$\overline{A}_n=4A_n+B_n=4A_n+A_{n+1}+A_{n-1}=4A_n+A_{n+1}+A_{n+1}-A_n=3A_n+2(A_{n+2}-A_n)=2A_{n+2}+A_n.$$

$$\overline{A}_n + A_n = 2B_{n+1}. \tag{11}$$

Proof: Using (10), $\overline{A}_n + A_n = 2A_{n+2} + 2A_n = 2B_{n+1}$.

$$2A_{n-2} + A_n = 11\underline{A}_n. \tag{12}$$

Proof: From (4),

$$\begin{aligned} 11\underline{A}_n &= 4A_n - B_n = 4A_n - (A_{n+1} + A_{n-1}) = 4A_n - (A_n + 2A_{n-1}) \\ &= 3A_n - 2(A_n - A_{n-2}) = 2A_{n-2} + A_n. \end{aligned}$$

$$11\underline{A}_n + A_n = 2B_{n-1}. \tag{13}$$

Proof: Again using (4), $11\underline{A}_n + A_n = 5A_n - B_n = -B_n + B_{n+1} + B_{n-1} = 2B_{n-1}$.

Now, by inspection of (9)-(13), we see that the following is an 11-term A.P. of the required form, with c.d. = $A_n = A_n(m)$:

$$\begin{gathered} (-2B_{n-1}(m), -11A_n(m-1), -2A_{n-2}(m), A_{n-3}(m), 2A_{n-1}(m), B_n(m), 2A_{n+1}(m), \\ A_{n+3}(m), 2A_{n+2}(m), A_n(m+1), 2B_{n+1}(m)). \end{gathered} \tag{14}$$

It only remains to show that, for some fixed m and n, this A.P. reduces to an 11-tuple of 2's. We find that setting $m = n = 0$ accomplishes this; for, in that case, the c.d. is $A_0(0) = F_0 = 0$, and one term, e.g., $B_0(0)$, is equal to $L_0 = 2$. Therefore, (14) is a valid solution of part (A).

Solution of Part (B)

Using parts (B), (A4)(iii), and (A4)(i) of [1], the following pair of 4-term A.P.'s are found, with c.d. = A_n and with terms of the required form:

$$(-3A_{n-2}, -A_{n-4}, B_{n-2}, 3A_{n-1}); \tag{15}$$

$$(3A_{n+1}, B_{n+2}, A_{n+4}, 3A_{n+2}). \tag{16}$$

Our goal, if possible, is to affix two additional terms of the required form between the two 4-term A.P.'s above, thereby forming a 10-term A.P. which satisfies the condition that, for some fixed m and n, all 10 terms equal 3. We require a few additional identities:

$$3A_{n-1} + A_n = 11\underline{A}_{n+1}. \tag{17}$$

Proof: Replacing n by $n+1$ in (12), we have:

$$11\underline{A}_{n+1} = 2A_{n-1} + A_{n+1} = 2A_{n-1} + A_n + A_{n-1} = A_n + 3A_{n-1}.$$

$$3A_{n+1} - A_n = \overline{A}_{n-1}. \tag{18}$$

Proof: Replacing n by $n-1$ in (10), we have:

$$\overline{A}_{n-1} = 2A_{n+1} + A_{n-1} = 2A_{n+1} + A_{n+1} - A_n = 3A_{n+1} - A_n.$$

$$\overline{A}_{n-1} - A_n = 11\underline{A}_{n+1}. \tag{19}$$

Proof: By (18),

$$\overline{A}_{n-1} - A_n = 3A_{n+1} - 2A_n = 3A_{n+1} - 2(A_{n+1} - A_{n-1}) = A_{n+1} + 2A_{n-1} = 11\underline{A}_{n+1}$$

[using (12), with $n+1$ replacing n]. By inspection of (15)-(19), we see that we have "bridged the gap" between the two 4-term A.P.'s, as required, producing an A.P. of 10 terms of the required form, with c.d. = A_n; this is given as follows:

$$\begin{aligned}(-3A_{n-2}(m), -A_{n-4}(m), B_{n-2}(m), 3A_{n-1}(m), 11A_{n+1}(m-1),\\ A_{n-1}(m+1), 3A_{n+1}(m), B_{n+2}(m), A_{n+4}(m), 3A_{n+2}(m)).\end{aligned} \tag{20}$$

Again setting $m=n=0$, the c.d. is 0 in this case, and one term, e.g., $3A_1(0)=3F_1=3$; thus, in this case, we have a 10-tuple of 3's, as required. This shows that the 10-tuple in (20) provides a solution to part (B).

Solution of Part (C)

Using parts (B) and (A2) of [1], we find the following 6-term A.P. of the required form, with c.d. = A_n:

$$(-B_{n-1}, -A_{n-2}, A_{n-1}, A_{n+1}, A_{n+2}, B_{n+1}). \tag{21}$$

Our goal, if possible, is to affix two terms to this A.P. (at either end or at each end), which are of the required form and satisfy the desired conditions. We require two additional identities:

$$B_{n+1}+A_n = 11\underline{A}_{n+2}. \tag{22}$$

Proof: Replacing n by $n+2$ in (12), we have:

$$11\underline{A}_{n+2} = 2A_n + A_{n+2} = A_n + (A_n + A_{n+2}) = A_n + B_{n+1}.$$

$$B_{n-1}+A_n = \overline{A}_{n-2}. \tag{23}$$

Proof: Replacing n by $n-2$ in (10), we have:

$$\overline{A}_{n-2} = 2A_n + A_{n-2} = A_n + (A_n + A_{n-2}) = A_n + B_{n-1}.$$

By inspection of (21)-(23), we see that the following 8-term A.P. has c.d. = A_n:

$$(-A_{n-2}(m+1), -B_{n-1}(m), -A_{n-2}(m), A_{n-1}(m), A_{n+1}(m), A_{n+2}(m), B_{n+1}(m), 11A_{n+2}(m-1)). \tag{24}$$

Again setting $m=n=0$, we see that the c.d. = 0 and each term, e.g., $A_1(0)=F_1=1$ for this case; thus, for this case, we obtain an 8-tuple of 1's. This shows that (24) yields a solution to part (C) and we are done.

Also solved by the proposer.