SEQUENCES RELATED TO AN INFINITE PRODUCT EXPANSION FOR THE SQUARE ROOT AND CUBE ROOT FUNCTIONS

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In the first section we shall consider three sequences associated with the square root function. In the second section we shall consider three sequences associated with the cube root function. In the third section, after considering three different sequences associated with the square root function, we make comparisons with the hope (unfulfilled) of a possible generalization.

1. THE SQUARE ROOT FUNCTION

In [1], Eric Wingler showed that repeated use of the identity

$$\sqrt{1+r}=\frac{2r+2}{r+2}\sqrt{1+\frac{r^2}{4r+4}}$$

leads to an infinite product expansion of $\sqrt{1+r}$ in the following manner: For $a_1>-1$ and n a positive integer, defining

$$a_{n+1}=\frac{a_n^2}{4a_n+4} \quad\text{and}\quad b_n=\frac{2a_n+2}{a_n+2}$$

implies $\sqrt{1+a_1}=\prod_{i=1}^{\infty}b_i$.

In the sequel, n will always denote a positive integer and, *a propos* the preceding product, for $n\geq 1$, define $c_n=b_1b_2b_3\dots b_n$.

In Definition 1 we shall define three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, which will depend on a_1 and which are related to $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. These definitions are motivated by our desire to have, when a_1 is a positive integer, x_n, y_n, and z_n be integers such that $c_n=x_n/y_n$, $(x_n,y_n)=1$, and z_n is the numerator of a_{n+1} when it is written as a reduced fraction with positive numerator. As can be seen from Theorem 2 and Lemma 3, these definitions will give us even more than what we desire.

Definition 1: Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ as follows:

For $2|a_1$, define

$$x_1=a_1+1,\ y_1=\frac{1}{2}a_1+1,\ \text{and}\ z_1=\left(\frac{a_1}{2}\right)^2;$$

otherwise,

$$x_1=2a_1+2,\ y_1=a_1+2,\ \text{and}\ z_1=a_1^2.$$

For $4|a_1$ and $n\geq 1$, define

$$x_{n+1}=x_ny_n,\quad y_{n+1}=y_n^2-\frac{z_n}{2},\ \text{and}\ z_{n+1}=\left(\frac{z_n}{2}\right)^2;$$

otherwise,

$$x_{n+1} = 2x_n y_n, \quad y_{n+1} = 2y_n^2 - z_n, \quad \text{and} \quad z_{n+1} = z_n^2.$$

As an example, for $a_1 = 6$, we have that the first five terms of each of our six sequences are:

$$\begin{array}{lllll}
a_1 = 6 & a_2 = \frac{9}{7} & a_3 = \frac{81}{448} & a_4 = \frac{6561}{947968} & a_5 = \frac{43046721}{3619451788288} \\
b_1 = \frac{7}{4} & b_2 = \frac{32}{23} & b_3 = \frac{1058}{977} & b_4 = \frac{1909058}{1902497} & b_5 = \frac{7238989670018}{7238946623297} \\
c_1 = \frac{7}{4} & c_2 = \frac{56}{23} & c_3 = \frac{2576}{977} & c_4 = \frac{5033504}{1902497} & c_5 = \frac{19152452518976}{7238946623297} \\
x_1 = 7 & x_2 = 56 & x_3 = 2576 & x_4 = 5033504 & x_5 = 19152452518976 \\
y_1 = 4 & y_2 = 23 & y_3 = 977 & y_4 = 1902497 & y_5 = 7238946623297 \\
z_1 = 9 & z_2 = 81 & z_3 = 6561 & z_4 = 43046721 & z_5 = 1853020188851841.
\end{array}$$

We also have that $a_6 = \frac{1853020188851841}{5240234821309001823429836 8}$.

By Definition 1, for a_1 not an integer, the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are defined by:

$$x_1 = 2a_1 + 2, \quad y_1 = a_1 + 2, \quad \text{and} \quad z_1 = a_1^2$$

and, for $n \geq 1$,

$$x_{n+1} = 2x_n y_n, \quad y_{n+1} = 2y_n^2 - z_n, \quad \text{and} \quad z_{n+1} = z_n^2.$$

The main results, namely, Lemmas 3-6 and Corollary 7, do not require a_1 to be an integer. In fact, the only results for the square root function that do not hold when a_1 is not an integer are, not surprisingly, the ones relating to x_n, y_n, and z_n being relatively prime (Lemmas 8-10).

In Theorem 2 we shall state our results concerning the square root function. These results relate the six sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$.

Theorem 2: Let a_1 and n be integers such that $n \geq 1$ and $a_1 > -1$. We have that

$$a_{n+1} = \frac{z_n}{y_n^2 - z_n}, \quad b_{n+1} = \frac{x_{n+1} y_n}{x_n y_{n+1}}, \quad \text{and} \quad c_n = \frac{x_n}{y_n}.$$

In addition, depending on whether $4|a_1$ or not,

$$b_{n+1} = \frac{y_n^2}{y_{n+1}} \quad \text{or} \quad b_{n+1} = \frac{2y_n^2}{y_{n+1}}.$$

For a_1 an integer, we also have that

$$(z_n, y_n^2 - z_n) = 1, \quad (x_n, y_n) = 1, \quad \text{and} \quad (2y_n^2, y_{n+1}) = 1.$$

With Definition 1 as made, the proof of Theorem 2 is fairly straightforward and follows from Lemmas 3-6 and 8-10.

Lemma 3: For $n \geq 1$, $x_n^2 - (a_1 + 1)y_n^2 = -(a_1 + 1)z_n$.

Proof: This result is easily shown to be true for $n = 1$. Thus, assume this result is true for $n = k$, where $k \geq 1$. We shall prove this result is true for $n = k + 1$ in the case where 4 does not divide a_1. The proof is similar for $4|a_1$.

We have that

$$\begin{aligned}x_{k+1}^2-(a_1+1)y_{k+1}^2 &= (2x_ky_k)^2-(a_1+1)(2y_k^2-z_k)^2\\ &= 4x_k^2y_k^2-4(a_1+1)y_k^4+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2\\ &= 4y_k^2[x_k^2-(a_1+1)y_k^2]+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2\\ &= -4y_k^2(a_1+1)z_k+4(a_1+1)y_k^2z_k-(a_1+1)z_k^2=-(a_1+1)z_{k+1}. \quad \square\end{aligned}$$

Comment: Let a_1 be an integer such that a_1+1 is a perfect square. Since, by Definition 1, z_n is also a perfect square, we can let

$$k_n^2=(a_1+1)\frac{x_n^2}{(a_1+1)^2}=\frac{x_n^2}{a_1+1} \quad \text{and} \quad p_n^2=z_n.$$

Thus, by Lemma 3, $y_n^2=p_n^2+k_n^2$.

For $a_1=8$ and $n=1$, 2, 3, and 4, the identity $y_n^2=p_n^2+k_n^2$ gives us

$$\begin{aligned}5^2&=4^2+3^2\\ 17^2&=8^2+15^2\\ 257^2&=32^2+255^2\\ 65537^2&=512^2+65535^2.\end{aligned}$$

In this example, y_n is the n^{th} Fermat number.

Lemma 4: For $n\ge 1$ and $a_1>-1$, we have that $a_{n+1}=z_n/(y_n^2-z_n)$.

Proof: This result is easily shown to be true for $n=1$. Assume $a_{k+1}=z_k/(y_k^2-z_k)$, where $k\ge 1$. We shall prove this result is true for $n=k+1$ in the case where 4 does not divide a_1. The proof is similar for $4|a_1$.

Since

$$(y_k^2-z_k)^2a_{k+1}^2=z_k^2=z_{k+1}$$

and

$$\begin{aligned}(y_k^2-z_k)^2(4a_{k+1}+4) &= 4(y_k^2-z_k)(y_k^2-z_k)(a_{k+1}+1)\\ &= 4(y_k^2-z_k)[z_k+(y_k^2-z_k)]\\ &= 4(y_k^2-z_k)y_k^2\\ &= 4y_k^4-4y_k^2z_k+z_k^2-z_k^2\\ &= (2y_k^2-z_k)^2-z_k^2\\ &= y_{k+1}^2-z_{k+1}\end{aligned}$$

we see that

$$a_{k+2}=\frac{a_{k+1}^2}{4a_{k+1}+4}=\frac{(y_k^2-z_k)^2a_{k+1}^2}{(y_k^2-z_k)^2(4a_{k+1}+4)}=\frac{z_{k+1}}{y_{k+1}^2-z_{k+1}}. \quad \square$$

Lemma 5: For $n\ge 1$ and $a_1>-1$, we have that $b_{n+1}=x_{n+1}y_n/x_ny_{n+1}$. Also, for $4|a_1$, we have $b_{n+1}=y_n^2/y_{n+1}$; otherwise, $b_{n+1}=2y_n^2/y_{n+1}$.

Proof: By Lemma 4

$$b_{n+1}=\frac{2a_{n+1}+2}{a_{n+1}+2}=\frac{2y_n^2}{2y_n^2-z_n}.$$

Thus, for $4|a_1$,

$$b_{n+1}=\frac{2y_n^2}{2y_n^2-z_n}=\frac{y_n^2}{y_{n+1}}=\frac{x_ny_n^2}{x_ny_{n+1}}=\frac{x_{n+1}y_n}{x_ny_{n+1}};$$

otherwise,

$$b_{n+1}=\frac{2y_n^2}{2y_n^2-z_n}=\frac{2y_n^2}{y_{n+1}}=\frac{2x_ny_n^2}{x_ny_{n+1}}=\frac{x_{n+1}y_n}{x_ny_{n+1}}. \quad \square$$

Lemma 6: For $n\geq 1$ and $a_1>-1$, we have that $c_n=x_n/y_n$.

Proof: We easily see that

$$c_1=b_1=\frac{2a_1+2}{a_1+2}=\frac{x_1}{y_1}.$$

Now assume, for $k\geq 1$, that $c_k=x_k/y_k$. Thus, by Lemma 5,

$$c_{k+1}=c_kb_{k+1}=\frac{x_k}{y_k}\cdot\frac{x_{k+1}y_k}{x_ky_{k+1}}=\frac{x_{k+1}}{y_{k+1}}. \quad \square$$

As a corollary to Lemmas 4, 3, and 6, we have

Corollary 7: For $n\geq 1$ and $a_1>-1$, we have that $a_{n+1}=\frac{a_1+1}{c_n^2}-1$.

Proof: We have that

$$a_{n+1}=\frac{z_n}{y_n^2-z_n}=\frac{(a_1+1)z_n}{(a_1+1)(y_n^2-z_n)}=\frac{(a_1+1)y_n^2-x_n^2}{x_n^2}$$
$$=(a_1+1)\left(\frac{y_n}{x_n}\right)^2-1=\frac{a_1+1}{c_n^2}-1. \quad \square$$

The next lemma follows directly from Definition 1.

Lemma 8: For a_1 and n integers such that $n\geq 1$, exactly one of x_n, y_n, and z_n is even. More explicitly, we have that

when $a_1\equiv 0 \pmod 4$, z_n is even,
when $a_1\equiv 2 \pmod 4$, y_1 is even and, for $n\geq 2$, x_n is even,
when $a_1\equiv 1 \pmod 2$, x_n is even.

Lemma 9: For a_1 and n integers with $n\geq 1$, each of (y_n, z_n), (y_n, y_{n+1}), and (x_n, y_n) is a power of 2.

Proof: By Definition 1, $(y_1, z_1)=1=2^0$. We shall complete the proof by mathematical induction; thus, we shall also assume (y_k, z_k) is a power of 2, where $k\geq 1$. Also assume there is an odd prime p that divides (y_{k+1}, z_{k+1}). Since p divides z_{k+1} and $z_{k+1}|z_k^2$, we must have $p|z_k$. Now either

$$2y_{k+1} = 2y_k^2 - z_k \text{ or } y_{k+1} = 2y_k^2 - z_k.$$

Hence, since p is an odd prime such that $p|y_{k+1}$, and $p|z_k$, we see that $p|y_k$. Thus, p divides (y_k, z_k). This contradicts (y_k, z_k) being a power of 2.

Using the fact that, for $n \geq 1$, (y_n, z_n) is a power of 2, we shall now give indirect proofs that (y_n, y_{n+1}) and (x_n, y_n) are also powers of 2.

Thus, assume p is an odd prime that divides (y_n, y_{n+1}). Now

$$2y_{n+1} - 2y_n^2 = -z_n \text{ or } y_{n+1} - 2y_n^2 = -z_n.$$

In either case, $p|z_n$. Thus, p is an odd prime dividing (y_n, z_n); this is a contradiction.

Finally, assume p is an odd prime dividing (x_n, y_n). Thus, by Lemma 3, p divides

$$x_n\left(\frac{x_n}{a_1+1}\right) - y_n^2 = -z_n.$$

Thus, p is an odd prime dividing (y_n, z_n); this is a contradiction. □

Lemma 10: For a_1 and n integers such that $n \geq 1$, we have that

$$(z_n, y_n^2 - z_n) = 1, \ (2y_n^2, y_{n+1}) = 1, \text{ and } (x_n, y_n) = 1.$$

Proof: First notice that, by the preceding two lemmas,

$$(y_n, z_n) = 1, \ (y_n, y_{n+1}) = 1, \text{ and } (x_n, y_n) = 1.$$

Thus,

$$(z_n, y_n^2 - z_n) = (z_n, y_n^2) = 1$$

and, since y_{n+1} is an odd integer,

$$(2y_n^2, y_{n+1}) = (y_n^2, y_{n+1}) = 1. \ \square$$

2. THE CUBE ROOT FUNCTION

In [1], Eric Wingler also showed that repeated use of the identity

$$\sqrt[3]{1+s} = \frac{2s+3}{s+3}\sqrt[3]{1+\frac{2s^3+s^4}{(2s+3)^3}}$$

leads to an infinite product expansion of $\sqrt[3]{1+s}$ in the following manner: For $a_1 > 0$ and n a positive integer, defining

$$d_1 = a_1, \ d_{n+1} = \frac{2d_n^3 + d_n^4}{(2d_n+3)^3}, \text{ and } e_n = \frac{2d_n+3}{d_n+3},$$

implies $\sqrt[3]{1+d_1} = \prod_{i=1}^{\infty} e_i$.

A propos the preceding product, for $n \geq 1$, let $f_n = e_1e_2e_3 \ldots e_n$.

In Definition 11, we shall define three sequences, $\{u_n\}, \{v_n\}$, and $\{w_n\}$, which will depend on a_1 and which are related to $\{d_n\}, \{e_n\}$, and $\{f_n\}$. These definitions are motivated by our desire to have, when a_1 is a positive integer, u_n, v_n, and w_n be integers such that $f_n = u_n / v_n$ and w_n can be a numerator of d_{n+1} when it is written as a fraction; we do not require the fractions to be written in lowest terms. As can be seen in Theorem 12, which does not require a_1 to be an integer, the definitions in Definition 11 will give us even more than we desire.

Definition 11: Define the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$ as follows:

$$u_1 = 2a_1 + 3, \quad v_1 = a_1 + 3, \quad \text{and} \quad w_1 = a_1^4 + 2a_1^3,$$

and, for $n \geq 1$, define

$$u_{n+1} = u_n(3u_n^3 + 2w_n), \quad v_{n+1} = v_n(3u_n^3 + w_n), \quad \text{and} \quad w_{n+1} = w_n^3(2u_n^3 + w_n).$$

For a_1 an integer, the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are integer sequences.

In Theorem 12, we shall state our results concerning the cube root function. These results relate the six sequences $\{d_n\}, \{e_n\}, \{f_n\}, \{u_n\}, \{v_n\}$, and $\{w_n\}$.

Theorem 12: For $n \geq 1$,

$$d_{n+1} = \frac{w_n}{u_n^3}, \quad e_{n+1} = \frac{u_{n+1}v_n}{u_n v_{n+1}}, \quad \text{and} \quad f_n = \frac{u_n}{v_n}.$$

We also have that

$$e_{n+1} = \frac{3u_n^3 + 2w_n}{3u_n^3 + w_n}.$$

We shall now prove four lemmas and a corollary. These five results are analogous (also see the comment at the beginning of Section 3) to Lemmas 3-6 and Corollary 7. The four lemmas will provide a proof of Theorem 12.

Lemma 13: For $n \geq 1$, $u_n^3 - (a_1 + 1)v_n^3 = -w_n$.

Proof: This lemma is true for $n = 1$. Assuming this lemma is true for $n = k$, we see that

$$\begin{aligned} u_{k+1}^3 - (a_1+1)v_{k+1}^3 &= u_k^3(3u_k^3 + 2w_k)^3 - (a_1+1)v_k^3(3u_k^3 + w_k)^3 \\ &= u_k^3(3u_k^3 + 2w_k)^3 - (u_k^3 + w_k)(3u_k^3 + w_k)^3 \\ &= u_k^3(27u_k^9 + 54u_k^6 w_k + 36u_k^3 w_k^2 + 8w_k^3) \\ &\quad - (u_k^3 + w_k)(27u_k^9 + 27u_k^6 w_k + 9u_k^3 w_k^2 + w_k^3) \\ &= u_k^3(27u_k^6 w_k + 27u_k^3 w_k^2 + 7w_k^3) - w_k(27u_k^9 + 27u_k^6 w_k + 9u_k^3 w_k^2 + w_k^3) \\ &= -2u_k^3 w_k^3 - w_k^4 = -w_{k+1}. \quad \square \end{aligned}$$

Lemma 14: For $n \geq 1$ and $a_1 > -3/2$, $d_{n+1} = w_n / u_n^3$.

Proof: This result is easily seen to be true for $n = 1$. Thus, assume that, for $k \geq 1$, $d_{k+1} = w_k / u_k^3$. Since

$$2d_{k+1}^3 + d_{k+1}^4 = d_{k+1}^3(d_{k+1}+2) = \frac{w_k^3}{u_k^9} \cdot \frac{2u_k^3 + w_k}{u_k^3} = \frac{w_{k+1}}{u_k^{12}}$$

and

$$2d_{k+1} + 3 = \frac{3u_k^3 + 2w_k}{u_k^3} = \frac{u_k(3u_k^3 + 2w_k)}{u_k^4} = \frac{u_{k+1}}{u_k^4},$$

we have that

$$d_{k+2} = \frac{2d_{k+1}^3 + d_{k+1}^4}{(2d_{k+1}+3)^3} = \frac{w_{k+1}}{u_k^{12}} \cdot \frac{u_k^{12}}{u_{k+1}^3} = \frac{w_{k+1}}{u_{k+1}^3}. \quad \square$$

Lemma 15: For $n \geq 1$ and $a_1 > -3/2$,

$$\frac{3u_n^3 + 2w_n}{3u_n^3 + w_n} = e_{n+1} = \frac{u_{n+1}v_n}{u_n v_{n+1}}.$$

Proof: Let $n \geq 1$. By Lemma 14,

$$e_{n+1} = \frac{2d_{n+1}+3}{d_{n+1}+3} = \frac{3u_n^3 + 2w_n}{u_n^3} \cdot \frac{u_n^3}{3u_n^3 + w_n} = \frac{3u_n^3 + 2w_n}{3u_n^3 + w_n}.$$

By Definition 11, this implies that

$$e_{n+1} = \frac{u_n v_n(3u_n^3 + 2w_n)}{u_n v_n(3u_n^3 + w_n)} = \frac{u_{n+1}v_n}{u_n v_{n+1}}. \quad \square$$

Lemma 16: For $n \geq 1$ and $a_1 > -3/2$, $f_n = u_n / v_n$.

Proof: Since $u_1 = 2d_1 + 3$ and $v_1 = d_1 + 3$,

$$f_1 = e_1 = \frac{2d_1+3}{d_1+3} = \frac{2a_1+3}{a_1+3} = \frac{u_1}{v_1}.$$

Now assume that, for $k \geq 1$, $f_k = u_k / v_k$. Thus,

$$f_{k+1} = f_k e_{k+1} = \frac{u_k}{v_k} \cdot \frac{u_{k+1}v_k}{u_k v_{k+1}} = \frac{u_{k+1}}{v_{k+1}}. \quad \square$$

Corollary 17: For $n \geq 1$ and $a_1 > -3/2$, we have that

$$d_{n+1} = \frac{a_1+1}{f_n^3} - 1.$$

Proof: We have, by Lemmas 14, 13, and 16,

$$d_{n+1} = \frac{w_n}{u_n^3} = \frac{(a_1+1)v_n^3 - u_n^3}{u_n^3} = (a_1+1)\left(\frac{v_n}{u_n}\right)^3 - 1 = \frac{a_1+1}{f_n^3} - 1. \quad \square$$

3. COMPARING THE SEQUENCES ASSOCIATED WITH THE SQUARE ROOT AND CUBE ROOT FUNCTIONS

Comparing Definition 1 with a_1 not being an even integer and Definition 11, we have, for $n \geq 1$,

$$x_{n+1} = 2x_n y_n, \; y_{n+1} = 2y_n^2 - z_n, \text{ and } z_{n+1} = z_n^2,$$

but

$$u_{n+1} = u_n(3u_n^3 + 2w_n), \; v_{n+1} = v_n(3u_n^3 + w_n), \text{ and } w_{n+1} = w_n^3(2u_n^3 + w_n).$$

This does not lead to any obvious generalization.

Recall that one of the reasons for our choice of the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ was to have $(x_n, y_n) = 1$. When choosing the sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$, to make our task less difficult, we did not require that $(u_n, v_n) = 1$. If, for the square root function, we relax the relatively prime requirement, we can define three sequences that are associated with the square root function (compare Lemmas 3-6 with Lemmas 19-22) and which show more similarities with the three sequences we defined for the cube root function. We shall now define these three different sequences for the square root case.

Definition 18: Define the sequences $\{g_n\}, \{h_n\}$, and $\{j_n\}$ as follows:

$$g_1 = 2a_1 + 2, \; h_1 = a_1 + 2, \text{ and } j_1 = a_1^2(a_1 + 1),$$

and define, for $n \geq 1$,

$$g_{n+1} = g_n(2g_n^2 + 2j_n) = 2g_n(g_n^2 + j_n), \; h_{n+1} = h_n(2g_n^2 + j_n), \; j_{n+1} = j_n^2(g_n^2 + j_n).$$

We shall now verify four lemmas similar to Lemmas 3-6.

Lemma 19: For $n \geq 1$, $g_n^2 - (a_1 + 1)h_n^2 = -j_n$.

Proof: This result is easily shown to be true for $n = 1$. Thus, assume this result is true for $n = k$, where $k \geq 1$. We shall prove this result is true for $n = k + 1$. We have that

$$\begin{aligned} g_{k+1}^2 - (a_1+1)h_{k+1}^2 &= 4g_k^2(g_k^2 + j_k)^2 - (a_1+1)h_k^2(2g_k^2 - j_k)^2 \\ &= 4g_k^4[g_k^2 - (a_1+1)h_k^2] + 4g_k^4 j_k + 4g_k^2 j_k[g_k^2 - (a_1+1)h_k^2] \\ &\quad + 4g_k^2 j_k^2 - (a_1+1)h_k^2 j_k^2 \\ &= -4g_k^4 j_k + 4g_k^4 j_k - 4g_k^2 j_k^2 + 4g_k^2 j_k^2 - j_k^2(a_1+1)h_k^2 \\ &= -j_k^2(g_k^2 + j_k) = -j_{k+1}. \; \square \end{aligned}$$

Lemma 20: For $n \geq 1$ and $a_1 > -1$, we have that $a_{n+1} = j_n / g_n^2$.

Proof: This result is easily shown to be true for $n = 1$. Assume $a_{k+1} = j_k / g_k^2$, where $k \geq 1$. Now

$$a_{k+2} = \frac{a_{k+1}^2}{4a_{k+1} + 4} = \frac{j_k^2}{g_k^4} \cdot \frac{g_k^2}{4(g_k^2 + j_k)} = \frac{j_k^2}{4g_k^2(g_k^2 + j_k)} = \frac{j_k^2(g_k^2 + j_k)}{4g_k^2(g_k^2 + j_k)^2} = \frac{j_{k+1}}{g_{k+1}^2}. \; \square$$

Lemma 21: For $n \geq 1$ and $a_1 > -1$, we have that

$$\frac{2g_n^2 + 2j_n}{2g_n^2 + j_n} = b_{n+1} = \frac{g_{n+1}h_n}{g_n h_{n+1}}.$$

Proof: By Lemma 20,

$$b_{n+1} = \frac{2a_{n+1} + 2}{a_{n+1} + 2} = \frac{2(g_n^2 + j_n)}{g_n^2} \cdot \frac{g_n^2}{2g_n^2 + j_n} = \frac{2(g_n^2 + j_n)}{2g_n^2 + j_n} = \frac{2g_n(g_n^2 + j_n)h_n}{g_n h_n(2g_n^2 + j_n)} = \frac{g_{n+1}h_n}{g_n h_{n+1}}. \quad \square$$

Lemma 22: For $n \geq 1$ and $a_1 > -1$, we have that $c_n = g_n / h_n$.

Proof: This result is easily shown to be true for $n = 1$. Assume $c_k = g_k / h_k$. Thus, by Lemma 21,

$$c_{k+1} = c_k b_{k+1} = \frac{g_k}{h_k} \cdot \frac{g_{k+1}h_k}{g_k h_{k+1}} = \frac{g_{k+1}}{h_{k+1}}. \quad \square$$

Comparing Definitions 18 and 11 and Lemmas 19-22 with Lemmas 13-16, we see a very close connection between the square root function and the cube root function:

- $g_1 = 2a_1 + 2,\ \ h_1 = a_1 + 2,\ \ j_1 = a_1^2(a_1 + 1)$, and
  $u_1 = 2a_1 + 3,\ \ v_1 = a_1 + 3,\ \ w_1 = a_1^3(a_1 + 2)$

and, for $n \geq 1$ and $a_1 > -1$,

- $g_{n+1} = g_n(2g_n^2 + 2j_n),\ \ h_{n+1} = h_n(2g_n^2 + j_n),\ \ j_{n+1} = j_n^2(g_n^2 + j_n)$, and
  $u_{n+1} = u_n(3u_n^3 + 2w_n),\ \ v_{n+1} = v_n(3u_n^3 + w_n),\ \ w_{n+1} = w_n^3(2u_n^3 + w_n)$,
- $g_n^2 - (a_1 + 1)h_n^2 = -j_n$ and $u_n^3 - (a_1 + 1)v_n^3 = -w_n$,
- $a_{n+1} = \dfrac{j_n}{g_n^2}$ and $d_{n+1} = \dfrac{w_n}{u_n^3}$,
- $\dfrac{2g_n^2 + 2j_n}{2g_n^2 + j_n} = b_{n+1} = \dfrac{g_{n+1}h_n}{g_n h_{n+1}}$ and $\dfrac{3u_n^3 + 2w_n}{3u_n^3 + w_n} = e_{n+1} = \dfrac{u_{n+1}v_n}{u_n v_{n+1}}$,
- $c_n = \dfrac{g_n}{h_n}$ and $f_n = \dfrac{u_n}{v_n}$.

Sometimes the correct generalization, if any, and the obvious generalization, if any, are not quite exactly the same.

REFERENCE

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