SOME INFINITE SERIES SUMMATIONS USING POWER SERIES EVALUATED AT A MATRIX

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1. INTRODUCTION

In the notation of Horadam [5], write

$$W_n = W_n(a, b; p, q), \tag{1.1}$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \quad W_0 = a,\ W_1 = b,\ n \ge 2. \tag{1.2}$$

The sequence $\{W_n\}_{n=0}^{\infty}$ can be extended to negative subscripts by the use of (1.2) and, with this understanding, we simply write $\{W_n\}$.

The n^{th} terms of the well-known Fibonacci and Lucas sequences are then

$$\begin{cases} F_n = W_n(0, 1; 1, -1), \\ L_n = W_n(2, 1; 1, -1). \end{cases} \tag{1.3}$$

More generally, we write

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q), \end{cases} \tag{1.4}$$

where $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences, respectively, generated by (1.2). They have been studied extensively, particularly by Lucas [7].

The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.5}$$

$$V_n = \alpha^n + \beta^n, \tag{1.6}$$

where

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 - 4q}}{2} \tag{1.7}$$

are the roots, assumed distinct, of

$$x^2 - px + q = 0. \tag{1.8}$$

Write

$$\Delta = (\alpha - \beta)^2 = p^2 - 4q. \tag{1.9}$$

The Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.10}$$

has been studied widely in connection with the Fibonacci numbers and has the property

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \; n \text{ an integer (see [4]).} \tag{1.11}$$

Filipponi and Horadam [2] considered the matrix

$$Q_{k,x} = xQ^k = \begin{pmatrix} xF_{k+1} & xF_k \\ xF_k & xF_{k-1} \end{pmatrix}, \tag{1.12}$$

where x is an arbitrary real number and k is a nonnegative integer, and noted that

$$Q_{k,x}^n = \begin{pmatrix} x^n F_{kn+1} & x^n F_{kn} \\ x^n F_{kn} & x^n F_{kn-1} \end{pmatrix}. \tag{1.13}$$

Then they evaluated certain power series at the matrix $Q_{k,x}$ to obtain summation identities involving the Fibonacci and Lucas numbers. The identities had the following forms:

$$\sum_{n=0}^{\infty} a_n x^n F_{kn+1} = \frac{\phi_1 f(x\phi_1^k) - \phi_2 f(x\phi_2^k)}{\sqrt{5}}, \tag{1.14}$$

$$\sum_{n=0}^{\infty} a_n x^n F_{kn} = \frac{f(x\phi_1^k) - f(x\phi_2^k)}{\sqrt{5}}, \tag{1.15}$$

$$\sum_{n=0}^{\infty} a_n x^n F_{kn-1} = \frac{\phi_1 f(x\phi_2^k) - \phi_2 f(x\phi_1^k)}{\sqrt{5}}, \tag{1.16}$$

$$\sum_{n=0}^{\infty} a_n x^n L_{kn} = f(x\phi_1^k) + f(x\phi_2^k), \tag{1.17}$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1.18}$$

and

$$\phi_1 = \frac{1+\sqrt{5}}{2}, \quad \phi_2 = \frac{1-\sqrt{5}}{2}.$$

They also indicated how their procedures could be generalized to apply to $W_n(0, 1; p, -1)$ and $W_n(2, p; p, -1)$.

The object of this paper is to extend (1.14)-(1.17) to apply to the more general fundamental and primordial sequences of Lucas as defined in (1.4). Then, specializing to the Chebyshev polynomials of the first and second kinds, we obtain infinite series summations involving the sine and cosine functions that we believe are new.

2. THE MATRIX $A_{k,x}$

Define the matrix A by

$$A = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Then it can be shown by induction that

$$A^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}, \quad n \ge 0. \tag{2.2}$$

Associated with A, we define the matrix $A_{k,x}$ by

$$A_{k,x} = xA^k = \begin{pmatrix} xU_{k+1} & -x^n qU_{kn} \\ xU_k & -xqU_{k-1} \end{pmatrix}, \tag{2.3}$$

where x is an arbitrary real number and k is a nonnegative integer.

To prove the following lemma, we need to note that

$$V_k = U_{k+1} - qU_{k-1}, \tag{2.4}$$

$$U_k^2 - U_{k+1}U_{k-1} = q^{k-1}. \tag{2.5}$$

Each can be proved using Binet forms, and (2.5) is in fact a generalization of Simson's identity for Fibonacci numbers.

Lemma 1: The eigenvalues of $A_{k,x}$ are $x\alpha^k$ and $x\beta^k$.

Proof: Using (2.4) and (2.5), we see that the characteristic equation of $A_{k,x}$ simplifies to

$$t^2 - xV_k t + x^2 q^k = 0. \tag{2.6}$$

Recalling that $V_k = \alpha^k + \beta^k$ and $q = \alpha\beta$, we see, by substitution, that the eigenvalues are as stated. □

Another important property of $A_{k,x}$ is

$$A_{k,x}^n = (xA^k)^n = x^n A^{kn} = \begin{pmatrix} x^n U_{kn+1} & -xqU_{kn} \\ x^n U_{kn} & -x^n qU_{kn-1} \end{pmatrix}, \text{ by (2.2).} \tag{2.7}$$

The following is easily proved by induction:

$$\alpha^n = \alpha U_n - qU_{n-1}, \quad n \ge 0. \tag{2.8}$$

Of course, (2.8) remains valid if we replace α by β.

3. THE MAIN RESULT

Assuming that f as defined in (1.18) has a domain of convergence which includes $x\alpha^k$ and $x\beta^k$ we have, using (2.7),

$$f(A_{k,x}) = \sum_{n=0}^{\infty} a_n A_{k,x}^n = \begin{pmatrix} \sum_{n=0}^{\infty} a_n x^n U_{kn+1} & -q\sum_{n=0}^{\infty} a_n x^n U_{kn} \\ \sum_{n=0}^{\infty} a_n x^n U_{kn} & -q\sum_{n=0}^{\infty} a_n x^n U_{kn-1} \end{pmatrix}. \tag{3.1}$$

On the other hand, from the theory of matrices [3], it is known that $f(A_{k,x}) = c_0 I + c_1 A_{k,x}$, where I is the identity 2×2 matrix and where c_0 and c_1 can be obtained by solving

$$\begin{cases} c_0 + c_1 x\alpha^k = f(x\alpha^k), \\ c_0 + c_1 x\beta^k = f(x\beta^k). \end{cases}$$

That is,

$$f(A_{k,x}) = \left(\frac{x\alpha^k f(x\beta^k) - x\beta^k f(x\alpha^k)}{x(\alpha^k - \beta^k)}\right) I + \left(\frac{f(x\alpha^k) - f(x\beta^k)}{x(\alpha^k - \beta^k)}\right) A_{k,x}. \tag{3.2}$$

This is Sylvester's matrix interpolation formula [8]. Noting that $\alpha^k - \beta^k = \sqrt{\Delta}\, U_k$ and using (2.8), the right side of (3.2) can be simplified to yield

$$f(A_{k,x}) = \begin{pmatrix} \dfrac{\alpha f(x\alpha^k) - \beta f(x\beta^k)}{\sqrt{\Delta}} & \dfrac{q(f(x\beta^k) - f(x\alpha^k))}{\sqrt{\Delta}} \\ \dfrac{f(x\alpha^k) - f(x\beta^k)}{\sqrt{\Delta}} & \dfrac{\alpha f(x\beta^k) - \beta f(x\alpha^k)}{\sqrt{\Delta}} \end{pmatrix}. \tag{3.3}$$

These observations lead to our main result.

Theorem 1: If *f* as defined in (1.18) has a domain of convergence which includes $x\alpha^k$ and $x\beta^k$, then

$$\sum_{n=0}^{\infty} a_n x^n U_{kn+1} = \frac{\alpha f(x\alpha^k) - \beta f(x\beta^k)}{\sqrt{\Delta}}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn} = \frac{f(x\alpha^k) - f(x\beta^k)}{\sqrt{\Delta}}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a_n x^n U_{kn-1} = \frac{\beta f(x\alpha^k) - \alpha f(x\beta^k)}{q\sqrt{\Delta}}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a_n x^n V_{kn} = f(x\alpha^k) + f(x\beta^k). \tag{3.7}$$

We note that (3.4)-(3.6) are obtained by comparing (3.1) and (3.3). Identity (3.7) is obtained by using (2.4), (3.4), and (3.6).

It is easily seen that (3.4)-(3.7) generalize (1.14)-(1.17) and also (5.6)-(5.17) of [2]. In the next section we apply (3.5) and (3.7) to the Chebyshev polynomials and obtain infinite sums involving the sine and cosine functions.

4. APPLICATIONS

Let $\{T_n(x)\}_{n=0}^{\infty}$ and $\{S_n(x)\}_{n=0}^{\infty}$ denote the Chebyshev polynomials of the first and second kinds, respectively. Then

$$\left.\begin{aligned} S_n(x) &= \frac{\sin n\theta}{\sin\theta} \\ T_n(x) &= \cos n\theta \end{aligned}\right\}, \quad x = \cos\theta,\ n \geq 0.$$

Indeed $\{S_n(x)\}_{n=0}^{\infty}$ and $\{2T_n(x)\}_{n=0}^{\infty}$ are the fundamental and primordial sequences, respectively, generated by (1.2), where $p = 2\cos\theta$, $q = 1$. Thus,

$$\alpha = e^{i\theta} \quad \text{and} \quad \beta = e^{-i\theta}, \tag{4.1}$$

which are obtained by solving $t^2 - 2\cos\theta t + 1 = 0$. Further information about the Chebyshev polynomials can be found, for example, in [1] and [6].

To begin, we consider the following well-known power series each of which has the complex plane as its domain of convergence:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \tag{4.2}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{4.3}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \tag{4.4}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{4.5}$$

Now in (3.5), taking $U_n = \frac{\sin n\theta}{\sin\theta}$ and replacing f by the functions in (4.2)-(4.5), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \sin k(2n+1)\theta}{(2n+1)!} = \cos(x\cos k\theta)\sinh(x\sin k\theta), \tag{4.6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n} \sin 2kn\theta}{(2n)!} = \sin(x\cos k\theta)\sinh(x\sin k\theta), \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \sin k(2n+1)\theta}{(2n+1)!} = \sin(x\sin k\theta)\cosh(x\cos k\theta), \tag{4.8}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} \sin 2kn\theta}{(2n)!} = \sin(x\sin k\theta)\sinh(x\cos k\theta). \tag{4.9}$$

In (3.7), taking $V_n = 2\cos n\theta$ and replacing f by the functions in (4.2)-(4.5), we obtain, respectively,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} \cos k(2n+1)\theta}{(2n+1)!} = \sin(x\cos k\theta)\cosh(x\sin k\theta), \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cos 2kn\theta}{(2n)!} = \cos(x\cos k\theta)\cosh(x\sin k\theta), \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} \cos k(2n+1)\theta}{(2n+1)!} = \cos(x\sin k\theta)\sinh(x\cos k\theta), \tag{4.12}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}\cos 2kn\theta}{(2n)!}=\cos(x\sin k\theta)\cosh(x\cos k\theta). \tag{4.13}$$

At this point, we note that (4.6), (4.7), (4.10), and (4.11) generalize (40), (42), (41), and (43), respectively, of Walton [9].

As an example of the method, we prove (4.11).

Proof of (4.11): In (3.7), taking $V_n = 2\cos n\theta$ and $f(x)=\cos x$ we have, using (4.1) and (4.3),

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n} 2\cos 2kn\theta}{(2n)!} &= \cos(xe^{ik\theta})+\cos(xe^{-ik\theta})\\ &= 2\cos\left(x\left(\frac{e^{ik\theta}+e^{-ik\theta}}{2}\right)\right)\cos\left(x\left(\frac{e^{ik\theta}-e^{-ik\theta}}{2}\right)\right)\\ &= 2\cos(x\cos k\theta)\cos(ix\sin k\theta)\\ &= 2\cos(x\cos k\theta)\cosh(x\sin k\theta),\end{aligned}$$

which yields the result. □

We now obtain further interesting sums by employing some power series which occur in [1]. We restate them here for easy reference:

$$\log_e\left(1+\frac{z}{m}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{z^n}{m^n},\quad |z|<|m|, \tag{4.14}$$

$$\tan^{-1}\left(\frac{z}{m}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cdot\frac{z^{2n+1}}{m^{2n+1}},\quad |z|<|m|, \tag{4.15}$$

$$\sec\left(\frac{z}{m}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}\cdot\frac{z^{2n}}{m^{2n}},\quad |z|<\frac{\pi}{2}|m|, \tag{4.16}$$

$$\tan\left(\frac{z}{m}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad |z|<\frac{\pi}{2}|m|, \tag{4.17}$$

$$\operatorname{cosec}\left(\frac{z}{m}\right)-\frac{m}{z}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad 0<|z|<\pi|m|, \tag{4.18}$$

$$\cot\left(\frac{z}{m}\right)-\frac{m}{z}=\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}}{(2n)!}\cdot\frac{z^{2n-1}}{m^{2n-1}},\quad 0<|z|<\pi|m|. \tag{4.19}$$

Here, B_n and E_n are the Bernoulli and Euler numbers, respectively.

In (3.5), taking $U_n=\frac{\sin n\theta}{\sin\theta}$ and replacing f by the functions in (4.14)-(4.19) we obtain, respectively,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n\sin kn\theta}{nm^n}=\frac{1}{2i}\log_e\left(\frac{m+xe^{ik\theta}}{m+xe^{-ik\theta}}\right),\quad |x|<|m|, \tag{4.20}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\sin k(2n+1)\theta}{(2n+1)m^{2n+1}}=\frac{1}{2}\tanh^{-1}\left(\frac{2mx\sin k\theta}{m^2+x^2}\right),\quad |x|<|m|, \tag{4.21}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}x^{2n}\sin 2kn\theta}{(2n)!m^{2n}}=\frac{2\sin\left(\frac{x\cos k\theta}{m}\right)\sinh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|, \tag{4.22}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{\sinh\left(\frac{2x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|,\end{aligned} \tag{4.23}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{2\cos\left(\frac{x\cos k\theta}{m}\right)\sinh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)-\cosh\left(\frac{2x\sin k\theta}{m}\right)}+\frac{m\sin k\theta}{x},\quad 0<|x|<\pi|m|,\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}B_{2n}x^{2n-1}\sin k(2n-1)\theta}{(2n)!m^{2n-1}}\\&=\frac{\sinh\left(\frac{2x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)-\cosh\left(\frac{2x\sin k\theta}{m}\right)}+\frac{m\sin k\theta}{x},\quad 0<|x|<\pi|m|.\end{aligned} \tag{4.25}$$

As stated at the beginning of Section 3, the domains of validity are determined by the requirement that the eigenvalues, in this case $xe^{ik\theta}$ and $xe^{-ik\theta}$, must lie within the radius of convergence of the relevant power series. The proofs follow essentially the same lines as the proof of (4.11) demonstrated earlier, employing well-known properties of the relevant functions.

Finally in (3.7), taking $V_n=2\cos n\theta$ and replacing f by the functions in (4.14)-(4.19), we obtain, respectively,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n\cos kn\theta}{nm^n}=\frac{1}{2}\log_e\left(1+\frac{2x\cos k\theta}{m}+\frac{x^2}{m^2}\right),\quad |x|<|m|, \tag{4.26}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\cos k(2n+1)\theta}{(2n+1)m^{2n+1}}=\frac{1}{2}\tan^{-1}\left(\frac{2mx\cos k\theta}{m^2-x^2}\right),\quad |x|<|m|, \tag{4.27}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}x^{2n}\cos 2kn\theta}{(2n)!\,m^{2n}}=\frac{2\cos\left(\frac{x\cos k\theta}{m}\right)\cosh\left(\frac{x\sin k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|, \tag{4.28}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}$$
$$=\frac{\sin\left(\frac{2x\cos k\theta}{m}\right)}{\cos\left(\frac{2x\cos k\theta}{m}\right)+\cosh\left(\frac{2x\sin k\theta}{m}\right)},\quad |x|<\frac{\pi}{2}|m|, \tag{4.29}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}$$
$$=\frac{2\sin\left(\frac{x\cos k\theta}{m}\right)\cosh\left(\frac{x\sin k\theta}{m}\right)}{\cosh\left(\frac{2x\sin k\theta}{m}\right)-\cos\left(\frac{2x\cos k\theta}{m}\right)}-\frac{m\cos k\theta}{x},\quad 0<|x|<\pi|m|, \tag{4.30}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}2^{2n}B_{2n}x^{2n-1}\cos k(2n-1)\theta}{(2n)!\,m^{2n-1}}$$
$$=\frac{\sin\left(\frac{2x\cos k\theta}{m}\right)}{\cosh\left(\frac{2x\sin k\theta}{m}\right)-\cos\left(\frac{2x\cos k\theta}{m}\right)}-\frac{m\cos k\theta}{x},\quad 0<|x|<\pi|m|. \tag{4.31}$$

REFERENCES

1. M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. New York: Dover, 1972.
2. P. Filipponi & A. F. Horadam. "A Matrix Approach to Certain Identities." *The Fibonacci Quarterly* **26.2** (1988):115-26.
3. F. R. Gantmacher. *The Theory of Matrices*. New York: Chelsea, 1960.
4. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1980.
5. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* **3.2** (1965):161-76.
6. A. F. Horadam. "Tschebyscheff and Other Functions Associated with the Sequence $\{W_n(a,b;p,q)\}$." *The Fibonacci Quarterly* **7.1** (1969):14-22.
7. E. Lucas. *Théorie des Nombres*. Paris: Albert Blanchard, 1961.
8. H. W. Turnbull & A. C. Aitken. *An Introduction to the Theory of Canonical Matrices*. New York: Dover, 1961.
9. J. E. Walton. "Lucas Polynomials and Certain Circular Functions of Matrices." *The Fibonacci Quarterly* **14.1** (1976):83-87.

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