SOME SUMMATION IDENTITIES USING GENERALIZED Q-MATRICES

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1. INTRODUCTION

In a belated acknowledgment, Hoggatt [3] states:

> The first use of the Q-matrix to generate the Fibonacci numbers appears in an abstract of a paper by Professor J. L. Brenner by the title "Lucas' Matrix." This abstract appeared in the March 1951 *American Mathematical Monthly* on pages 221 and 222. The basic exploitation of the Q-matrix appeared in 1960 in the San Jose State College Master's thesis of Charles H. King with the title "Some Further Properties of the Fibonacci Numbers." Further utilization of the Q-matrix appears in the *Fibonacci Primer* sequence parts I-V.

For a comprehensive history of the Q-matrix, see Gould [2]. Numerous analogs of the Q-matrix relating to third-order recurrences have been used. See, for instance, Waddill and Sacks [13], Shannon and Horadam [10], and Waddill [11]. Mahon [8] has made extensive use of matrices to study his third-order diagonal functions of the Pell polynomials. Recently, Waddill [12] considered a general Q-matrix. He defined and used the $k \times k$ matrix

$$R = \begin{pmatrix} r_0 & r_1 & \cdots & r_{k-1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{pmatrix}$$

in relation to a k-order linear recursive sequence $\{V_n\}$, where

$$V_n = \sum_{i=0}^{k-1} r_i V_{n-1-i}, \quad n \geq k.$$

The matrix R generalized the matrix Q_r of Ivie [5].

In the notation of Horadam [4], write

$$W_n = W_n(a, b; p, q) \tag{1.1}$$

so that

$$W_n = pW_{n-1} - qW_{n-2}, \ W_0 = a, \ W_1 = b, \quad n \geq 2. \tag{1.2}$$

With this notation, define

$$\begin{cases} U_n = W_n(0, 1; p, q), \\ V_n = W_n(2, p; p, q). \end{cases} \tag{1.3}$$

Indeed, $\{U_n\}$ and $\{V_n\}$ are the fundamental and primordial sequences generated by (1.2). They have been studied extensively, particularly by Lucas [7]. Further information can be found in [1], [4], and [6].

The most commonly used matrix in relation to the recurrence relation (1.2) is

$$M = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix}, \tag{1.4}$$

which, for $p = -q = 1$, reduces to the ordinary Q-matrix. In this paper we define a more general matrix $M_{k,m}$ parametrized by k and m and reducing to M for $k = m = 1$. We use $M_{k,m}$ to develop various summation identities involving terms from the sequences $\{U_n\}$ and $\{V_n\}$.

Our work is a generalization of the work of Mahon and Horadam [9] who used several pairs of 2×2 matrices to generate summation identities involving terms from the Pell polynomial sequences

$$\begin{cases} P_n = W_n(0, 1; 2x, -1), \\ Q_n = W_n(2, 2x; 2x, -1). \end{cases} \tag{1.5}$$

We generalize their work in two ways. First, we consider sequences generated by a more general recurrence relation. Second, our parametrization of the matrix $M_{k,m}$ includes all the matrices considered by Mahon and Horadam as special cases.

2. THE MATRIX $M_{k,m}$

Before proceeding, we state some results which are used subsequently. None of these is new and each can be proved using Binet forms. If

$$\Delta = p^2 - 4q, \tag{2.1}$$

then

$$U_{n+1} - qU_{n-1} = V_n, \tag{2.2}$$

$$V_{n+1} - qV_{n-1} = \Delta U_n, \tag{2.3}$$

$$V_{2k} - 2q^k = \Delta U_k^2, \tag{2.4}$$

$$U_{k+m} - q^m U_{k-m} = U_m V_k, \tag{2.5}$$

$$V_{k+m} - q^m V_{k-m} = \Delta U_k U_m \tag{2.6}$$

$$U_{k+m} U_{k-m} - U_k^2 = -q^{k-m} U_m^2, \tag{2.7}$$

$$V_{k+m} V_{k-m} - V_k^2 = \Delta q^{k-m} U_m^2, \tag{2.8}$$

$$U_{n+m} U_{n_1+m} - q^m U_n U_{n_1} = U_m U_{n+n_1+m}. \tag{2.9}$$

By induction it can be proved that, for the matrix M in (1.4),

$$M^n = \begin{pmatrix} U_{n+1} & -qU_n \\ U_n & -qU_{n-1} \end{pmatrix}, \tag{2.10}$$

where n is an integer.

We now give a generalization of the matrix M. Associated with the recurrence (1.2) and with $\{U_n\}$ as in (1.3), define

$$M_{k,m} = \begin{pmatrix} U_{k+m} & -q^m U_k \\ U_k & -q^m U_{k-m} \end{pmatrix}, \tag{2.11}$$

where k and m are integers. By induction and making use of (2.9), it can be shown that, for all integral n,

$$M_{k,m}^n = U_m^{n-1} \begin{pmatrix} U_{nk+m} & -q^m U_{nk} \\ U_{nk} & -q^m U_{nk-m} \end{pmatrix}. \tag{2.12}$$

When $k = m = 1$, we see that $M_{k,m}$ reduces to M and $M_{k,m}^n$ reduces to M^n.

3. SUMMATION IDENTITIES

We now use the matrix $M_{k,m}$ to produce summation identities involving terms from $\{U_n\}$ and $\{V_n\}$. Using (2.5) and (2.7), we find that the characteristic equation of $M_{k,m}$ is

$$\lambda^2 - U_m V_k \lambda + q^k U_m^2 = 0 \tag{3.1}$$

and, by the Cayley-Hamilton theorem,

$$M_{k,m}^2 - U_m V_k M_{k,m} + q^k U_m^2 I = 0, \tag{3.2}$$

where I is the 2×2 unit matrix. From (3.2), we have

$$(U_m V_k M_{k,m} - q^k U_m^2 I)^n M_{k,m}^j = M_{k,m}^{2n+j}, \tag{3.3}$$

and expanding yields

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} U_m^{2n-i} V_k^i M_{k,m}^{i+j} = M_{k,m}^{2n+j}. \tag{3.4}$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} (-1)^{n-i} q^{k(n-i)} V_k^i U_{(i+j)k+m} = U_{(2n+j)k+m}. \tag{3.5}$$

Again from (3.2),

$$(M_{k,m}^2 + q^k U_m^2 I)^n = U_m^n V_k^n M_{k,m}^n, \tag{3.6}$$

and expanding we have

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_m^{2(n-i)} M_{k,m}^{2i} = U_m^n V_k^n M_{k,m}^n. \tag{3.7}$$

Using (2.12) to equate upper left entries gives

$$\sum_{i=0}^n \binom{n}{i} q^{k(n-i)} U_{2ik+m} = V_k^n U_{nk+m}. \tag{3.8}$$

Once again, from (3.2),

$$(M_{2k,m}-q^kU_mI)^2 = U_m(V_{2k}-2q^k)M_{2k,m} = \Delta U_mU_k^2M_{2k,m}, \tag{3.9}$$

and expanding, after taking n^{th} powers, we have

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i q^{k(2n-i)}U_m^{2n-i}M_{2k,m}^i = \Delta^nU_m^nU_k^{2n}M_{2k,m}^n. \tag{3.10}$$

Equating upper left entries yields

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i q^{k(2n-i)}U_{2ik+m} = \Delta^nU_k^{2n}U_{2nk+m}. \tag{3.11}$$

From (3.9),

$$(M_{2k,m}-q^kU_mI)^{2n+1} = \Delta^nU_m^nU_k^{2n}(M_{2k,m}^{n+1}-q^kU_mM_{2k,m}^n). \tag{3.12}$$

Equating upper left entries yields, after simplifying,

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1} q^{k(2n+1-i)}U_{2ik+m} = \Delta^nU_k^{2n}(U_{2(n+1)k+m}-q^kU_{2nk+m}), \tag{3.13}$$

and using (2.5) to simplify the right side gives

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1} q^{k(2n+1-i)}U_{2ik+m} = \Delta^nU_k^{2n+1}V_{(2n+1)k+m}. \tag{3.14}$$

This should be compared to (3.11).

Manipulating the characteristic equation (3.1), we have $(2\lambda-U_mV_k)^2 = \Delta U_m^2V_k^2$, so that

$$(2M_{k,m}-U_mV_kI)^{2n} = \Delta^nU_m^{2n}U_k^{2n}I. \tag{3.15}$$

Expanding gives

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i 2^iU_m^{2n-i}V_k^{2n-i}M_{k,m}^i = \Delta^nU_m^{2n}U_k^{2n}I. \tag{3.16}$$

Equating upper left entries and also lower left entries yields, respectively,

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i 2^iV_k^{2n-i}U_{ik+m} = \Delta^nU_k^{2n}U_m, \tag{3.17}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i 2^iV_k^{2n-i}U_{ik} = 0. \tag{3.18}$$

We note that (3.17) reduces to (3.18) when $m = 0$.

Multiplying both sides of (3.15) by $(2M_{k,m}-U_mV_kI)$ and expanding gives

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1} 2^iU_m^{2n+1-i}V_k^{2n+1-i}M_{k,m}^i = \Delta^nU_m^{2n}U_k^{2n}(2M_{k,m}-U_mV_kI). \tag{3.19}$$

Equating upper left entries yields

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}2^i V_k^{2n+1-i}U_{ik+m}=\Delta^n U_k^{2n+1}V_m, \tag{3.20}$$

which should be compared to (3.17).

Now, using (3.5), we have

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}q^{k(n-i)}V_k^i(U_{(i+j)k+m+1}-qU_{(i+j)k+m-1})=U_{(2n+j)k+m+1}-qU_{(2n+j)k+m-1},$$

and (2.2) shows that this simplifies to

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}q^{k(n-i)}V_k^iV_{(i+j)k+m}=V_{(2n+j)k+m}. \tag{3.21}$$

Making use of (2.2) and (2.3) and working in the same manner with identities (3.8), (3.11), (3.14), (3.17), and (3.20) yields, respectively,

$$\sum_{i=0}^{n}\binom{n}{i}q^{k(n-i)}V_{2ik+m}=V_k^nV_{nk+m}, \tag{3.22}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^iq^{k(2n-i)}V_{2ik+m}=\Delta^nU_k^{2n}V_{2nk+m}, \tag{3.23}$$

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}q^{k(2n+1-i)}V_{2ik+m}=\Delta^{n+1}U_k^{2n+1}U_{(2n+1)k+m}, \tag{3.24}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}(-1)^i2^iV_k^{2n-i}V_{ik+m}=\Delta^nU_k^{2n}V_m, \tag{3.25}$$

$$\sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^{i+1}2^iV_k^{2n+1-i}V_{ik+m}=\Delta^{n+1}U_k^{2n+1}U_m. \tag{3.26}$$

In what follows, we make use of the following result:

$$M_{k,m}^nM_{k_1,m}^{n_1}=U_m^{n+n_1-1}\begin{pmatrix}U_{nk+n_1k_1+m} & -q^mU_{nk+n_1k_1}\\ U_{nk+n_1k_1} & -q^mU_{nk+n_1k_1-m}\end{pmatrix}. \tag{3.27}$$

This is proved by multiplying the matrices on the left and using (2.9).

Consider now the special case of (3.2), where $k=m$. Then, using (2.5),

$$M_{k,k}^2=U_{2k}M_{k,k}-q^kU_k^2I. \tag{3.28}$$

Using (3.28) and (2.9), we can show by induction that, for $n\geq 2$,

$$M_{k,k}^n=U_k^{n-2}(U_{nk}M_{k,k}-q^kU_kU_{(n-1)k}I). \tag{3.29}$$

The binomial theorem applied to (3.29) gives

$$U_k^{(n-2)s}\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_k^{s-i}U_{(n-1)k}^{s-i}U_{nk}^{i}M_{k,k}^{i+j}=M_{k,k}^{ns+j} \tag{3.30}$$

Equating lower left entries of the relevant matrices then yields

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}U_{(i+j)k}=U_k^sU_{(ns+j)k}. \tag{3.31}$$

Multiplying both sides of (3.30) by $M_{k_1,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}U_{(i+j)k+k_1}=U_k^sU_{(ns+j)k+k_1}, \tag{3.32}$$

which generalizes (3.31).

Again from (3.29), after transposing terms and raising to a power s, we obtain

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^{(n-1)(s-i)}U_{(n-1)k}^{s-i}M_{k,k}^{ni}=U_k^{(n-2)s}U_{nk}^{s}M_{k,k}^{s}, \tag{3.33}$$

which yields

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^iU_{(n-1)k}^{s-i}U_{nik}=U_{nk}^sU_{sk}. \tag{3.34}$$

Multiplying both sides of (3.33) by $M_{k_1,k}$ and using (3.27) to equate lower left entries gives

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^iU_{(n-1)k}^{s-i}U_{nik+k_1}=U_{nk}^sU_{sk+k_1}, \tag{3.35}$$

which generalizes (3.34).

Continuing in this manner after yet again transposing terms in (3.29) and raising to a power s, we obtain

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^{(n-2)(s-i)}U_{nk}^{s-i}M_{k,k}^{(n-1)i+s}=q^{ks}U_k^{(n-1)s}U_{(n-1)k}^{s}I. \tag{3.36}$$

Equating upper left entries and lower left entries yields, respectively,

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s+1)k}=q^{ks}U_{(n-1)k}^{s}U_k, \tag{3.37}$$

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s)k}=0. \tag{3.38}$$

Multiplying (3.36) by $M_{k_1,k}$ and equating lower left entries yields

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^iU_k^iU_{nk}^{s-i}U_{((n-1)i+s)k+k_1}=q^{ks}U_{(n-1)k}^{s}U_{k_1}. \tag{3.39}$$

We note that, when $k_1 = k$, (3.39) reduces to (3.37) and when $k_1 = 0$, (3.39) reduces to (3.38).

Now, manipulating (3.32), (3.35), and (3.39) in the same way that (3.5) was manipulated to yield (3.21), we obtain, respectively,

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{k(s-i)}U_{(n-1)k}^{s-i}U_{nk}^{i}V_{(i+j)k+k_1} = U_k^s V_{(ns+j)k+k_1}, \tag{3.40}$$

$$\sum_{i=0}^{s}\binom{s}{i}q^{k(s-i)}U_k^i U_{(n-1)k}^{s-i}V_{nik+k_1} = U_{nk}^s V_{sk+k_1}, \tag{3.41}$$

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^i U_k^i U_{nk}^{s-i}V_{((n-1)i+s)k+k_1} = q^{ks}U_{(n-1)k}^s V_{k_1}. \tag{3.42}$$

4. THE MATRIX X_k

We have found a matrix having the property of generating terms from $\{U_n\}$ and $\{V_n\}$ simultaneously. It is a generalization of the matrix W introduced by Mahon and Horadam [9]. Define

$$X_k = \begin{pmatrix} V_k & U_k \\ \Delta U_k & V_k \end{pmatrix}, \; k \text{ an integer.} \tag{4.1}$$

Then by induction we have, for integral n,

$$X_k^n = 2^{n-1}\begin{pmatrix} V_{nk} & U_{nk} \\ \Delta U_{nk} & V_{nk} \end{pmatrix}. \tag{4.2}$$

Noting that $X_1^{m+n} = X_1^m \cdot X_1^n$ produces the well-known identities

$$2V_{m+n} = V_m V_n + \Delta U_m U_n, \tag{4.3}$$

$$2U_{m+n} = V_m U_n + U_m V_n. \tag{4.4}$$

The characteristic equation for X_k is

$$\lambda^2 - 2V_k\lambda + 4q^k = 0 \tag{4.5}$$

and so, by the Cayley-Hamilton theorem

$$X_k^2 - 2V_k X_k + 4q^k I = 0. \tag{4.6}$$

Using (4.3) and (4.4), we see that

$$X_k^n X_{k_1} = 2^n \begin{pmatrix} V_{nk+k_1} & U_{nk+k_1} \\ \Delta U_{nk+k_1} & V_{nk+k_1} \end{pmatrix}. \tag{4.7}$$

Considering the case $k = 1$, we can show by induction, with the aid of (4.6), that

$$X_1^n = 2^{n-1}(U_n X_1 - 2qU_{n-1}I), \quad n \geq 2, \tag{4.8}$$

which is analogous to (3.29).

It is interesting to note that the methods applied to $M_{k,m}$ when applied to X_k produce most of the summation identities that we have obtained so far. The exceptions are the identities that arose by using (3.29). The analogous procedure for X_k is to use (4.8), but the identities that arise are less general. For example, (4.8) produces

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}q^{s-i}U_{n-1}^{s-i}U_n^iU_{i+j+k_1}=U_{ns+j+k_1}, \tag{4.9}$$

which is a special case of (3.32).

5. THE MATRIX $N_{k,m}$

We have found yet another matrix defined in a similar manner to $M_{k,m}$ whose powers also generate terms of the sequences $\{U_n\}$ and $\{V_n\}$. Define

$$N_{k,m}=\begin{pmatrix} V_{k+m} & -q^mV_k \\ V_k & -q^mV_{k-m}\end{pmatrix}. \tag{5.1}$$

Then for all integral n,

$$N_{k,m}^{2n}=U_m^{2n-1}\Delta^n\begin{pmatrix} U_{2nk+m} & -q^mU_{2nk} \\ U_{2nk} & -q^mU_{2nk-m}\end{pmatrix}, \tag{5.2}$$

$$N_{k,m}^{2n-1}=U_m^{2n-2}\Delta^{n-1}\begin{pmatrix} V_{(2n-1)k+m} & -q^mV_{(2n-1)k} \\ V_{(2n-1)k} & -q^mV_{(2n-1)k-m}\end{pmatrix}. \tag{5.3}$$

The characteristic equation of $N_{k,m}$ is

$$\lambda^2-\Delta U_kU_m\lambda-\Delta q^kU_m^2=0, \tag{5.4}$$

and so

$$N_{k,m}^2-\Delta U_kU_mN_{k,m}-\Delta q^kU_m^2I=0. \tag{5.5}$$

Using the previous techniques and due to the manner in which powers of $N_{k,m}$ are defined, we have found some interesting summation identities. We note, however, that some of the methods applied to $M_{k,m}$ do not apply to $N_{k,m}$. For example, we could find no succinct counterpart to (3.29). We state only the essential details and omit summation identities that we have obtained previously.

Manipulating (5.5), we can write

$$\Delta U_m(U_kN_{k,m}+q^kU_mI)=N_{k,m}^2 \tag{5.6}$$

and

$$(2N_{k,m}-\Delta U_kU_mI)^2=\Delta U_m^2V_k^2I. \tag{5.7}$$

From (5.6) and (5.7), we have

$$\Delta^nU_m^n(U_kN_{k,m}+q^kU_mI)^n=N_{k,m}^{2n}, \tag{5.8}$$

$$(2N_{k,m}-\Delta U_k U_m I)^{2n}=\Delta^n U_m^{2n} V_k^{2n} I, \tag{5.9}$$

$$(2N_{k,m}-\Delta U_k U_m I)^{2n+1}=\Delta^n U_m^{2n} V_k^{2n}(2N_{k,m}-\Delta U_k U_m I). \tag{5.10}$$

Now expanding each of (5.8)-(5.10) and equating upper left entries of the relevant matrices leads, respectively, to

$$\sum_{\substack{i=0\\ i\text{ even}}}^{n}\binom{n}{i}q^{k(n-i)}\Delta^{\frac{i}{2}}U_k^i U_{ik+m}+\sum_{\substack{i=1\\ i\text{ odd}}}^{n}\binom{n}{i}q^{k(n-i)}\Delta^{\frac{i-1}{2}}U_k^i V_{ik+m}=U_{2nk+m}, \tag{5.11}$$

$$\sum_{\substack{i=0\\ i\text{ even}}}^{2n}\binom{2n}{i}2^i\Delta^{\frac{2n-i}{2}}U_k^{2n-i}U_{ik+m}-\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}\binom{2n}{i}2^i\Delta^{\frac{2n-1-i}{2}}U_k^{2n-i}V_{ik+m}=V_k^{2n}U_m, \tag{5.12}$$

$$\sum_{\substack{i=1\\ i\text{ odd}}}^{2n+1}\binom{2n+1}{i}2^i\Delta^{\frac{2n+1-i}{2}}U_k^{2n+1-i}V_{ik+m}-\sum_{\substack{i=0\\ i\text{ even}}}^{2n}\binom{2n+1}{i}2^i\Delta^{\frac{2n+2-i}{2}}U_k^{2n+1-i}U_{ik+m}=V_k^{2n+1}V_m. \tag{5.13}$$

Finally, making use of (2.2) and (2.3) and applying to (5.11)-(5.13) the same technique used to obtain (3.21), we have

$$\sum_{\substack{i=0\\ i\text{ even}}}^{n}\binom{n}{i}q^{k(n-i)}\Delta^{\frac{i}{2}}U_k^i V_{ik+m}+\sum_{\substack{i=1\\ i\text{ odd}}}^{n}\binom{n}{i}q^{k(n-i)}\Delta^{\frac{i+1}{2}}U_k^i U_{ik+m}=V_{2nk+m}, \tag{5.14}$$

$$\sum_{\substack{i=0\\ i\text{ even}}}^{2n}\binom{2n}{i}2^i\Delta^{\frac{2n-i}{2}}U_k^{2n-i}V_{ik+m}-\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}\binom{2n}{i}2^i\Delta^{\frac{2n+1-i}{2}}U_k^{2n-i}U_{ik+m}=V_k^{2n}V_m, \tag{5.15}$$

$$\sum_{\substack{i=1\\ i\text{ odd}}}^{2n+1}\binom{2n+1}{i}2^i\Delta^{\frac{2n+3-i}{2}}U_k^{2n+1-i}U_{ik+m}-\sum_{\substack{i=0\\ i\text{ even}}}^{2n}\binom{2n+1}{i}2^i\Delta^{\frac{2n+2-i}{2}}U_k^{2n+1-i}V_{ik+m}=\Delta V_k^{2n+1}U_m. \tag{5.16}$$

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