A COMBINATORIAL PROBLEM WITH A FIBONACCI SOLUTION

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1. THE PROBLEM

For many years I have enjoyed lecturing to groups of high school students about the excitement of mathematics. One diversion that never failed to capture their attention was as follows: Everyone was asked to *write down a three digit number* (*abc with* $a > c$), *reverse it* (*to form cba*), *find the difference* (*as a three-digit number*) *between the two numbers, and add the difference to its own reverse*. The amazed looks on the students' faces at discovering they had all reached the same *end number* 1089 was a sight to behold. The elementary algebra

$$\begin{array}{rccc} & a & b & c \\ - & c & b & a \\ \hline & a-c-1 & 9 & c-a+10 \\ + & c-a+10 & 9 & a-c-1 \\ \hline & 10 & 8 & 9 \\ \hline \end{array}$$

quickly explained the surprise. Finally, I would tease that the number 1089 is interesting in itself, being the square of 33 and its reverse being the square of 99.

The origins of the diversion are unknown to me, I learned of it from Rouse Ball ([1], p. 9). The question arises: *Can the diversion be extended to numbers other than those having three digits*? For two-digit numbers, the answer is yes, the end number always being 99. For four- and five-digit numbers, a little effort shows that *three* end numbers are possible, although the three numbers are different in the two cases. More effort is required to show that six- and seven-digit numbers give rise to different sets of *eight* possible end numbers. Thus, the sequence of the numbers of possible end numbers, corresponding to initial numbers of 2, 3, 4, 5, 6, 7, ... digits begins 1, 1, 3, 3, 8, 8, ... No prizes for guessing how it continues! Our main result is that the number of possible end numbers corresponding to initial numbers of $n+1$ digits is the Fibonacci number $F_{2[\frac{n+1}{2}]}$.

What of the end numbers themselves? The unique end number generated by two-digit numbers is 99, which turns out to be a divisor of *all* end numbers. The unique end number generated by three-digit numbers is 1089, which is 99×11. The three end numbers generated by four-digit numbers are 9999, 10890, and 10989, which are, respectively, 99×101, 99×110, and 99×111. These examples illustrate the general situation. With each $n+1$-digit number $X = x_n \ldots x_0$ ($x_n > x_0$) we associate an n-digit number $X^\flat$ consisting of a string of 0s and 1s, which has the property that the end number generated by X is $99X^\flat$. We give a simple characterization of the numbers $X^\flat$, and hence of the end numbers themselves.

2. THE CODE OF A NUMBER

Throughout our discussion, nonnegative integers will be written in decimal form and T will denote the number 10. We write $X = x_n \dots x_0$, where $x_0, \dots, x_n$ are integers between 0 and 9 inclusive, to denote the $n+1$-digit number $\sum_{i=0}^{n} x_i T^i$. The $n+1$-digit number obtained by reversing the digits of X is called the *reverse of* X (in $n+1$-digit arithmetic) and is denoted by X', whence $X' = x_0 \dots x_n$. Suppose that $X = x_n \dots x_0$ is such that $x_n > x_0$. Write the number $X - X'$ as an $n+1$-digit number—this may necessitate including some zeros at the front of the standard decimal representation of $X - X'$. Now reverse the digits of the difference $X - X'$ to obtain the number $(X - X')'$. Finally, add the difference to its reverse to produce the number X^* defined by the equation

$$X^* = X - X' + (X - X')'.$$

We wish to find the number, denoted here by a_n, of different (end numbers) X^* that are possible as X ranges over all $n+1$-digit numbers $x_n \dots x_0$ $(x_n > x_0)$. The diversion that motivated our discussion depends on the fact that $a_2 = 1$, i.e., for three-digit numbers, only one (end number) X^* can occur.

The key to our analysis is the association with each $n+1$-digit number $X = x_n \dots x_0$ $(x_n > x_0)$ an $n+1$-digit number $X^\sharp$ called the *code* of X. This code $X^\sharp$ comprises a string of 0s and 1s, has leading digit 1, final digit 0, and encodes all the information needed to pass from X to the (end number) X^* to which it gives rise. We first explain the construction of $X^\sharp$ informally, leaving a formal definition until later.

Write down the number $X = x_n \dots x_0$ $(x_n > x_0)$ and beneath it, its reverse $X' = x_0 \dots x_n$, as shown below:

$$\begin{array}{r} x_n \dots x_0 \\ - \ \underline{x_0 \dots x_n} \\ \underline{* \ \cdots \ *} \end{array}$$

Consider the role played by the i^{th} column from the right ($i = 0, \dots, n$) in the subtraction of X' from X. Define an integer z_i as follows: if a *ten* has to be *borrowed* from the $i+1^{\text{th}}$ column, z_i is 1; otherwise, it is 0. In this way we construct a string $z_0, \dots, z_n$ of 0s and 1s. The $n+1$-digit number $z_0 \dots z_n$ is called the *code* of X and is denoted by $X^\sharp$. Since we are assuming that $x_n > x_0$, $z_0 = 1$, and $z_n = 0$. The n-digit number $z_0 \dots z_{n-1}$ obtained by deleting the final 0 (z_n) from the code $z_0 \dots z_n$ of X is called the *truncated code* of X and is denoted by $X^\flat$.

To illustrate the above ideas, consider the six-digit number $X = 812311$. Subtracting X' from X, we find that

$$\begin{array}{r} 812311 \\ -\underline{113218} \\ \underline{699093} \end{array}$$

The columns for which a ten has to be borrowed from the adjacent column to the left are (labeling from the right) the $0^{\text{th}}, 1^{\text{st}}, 3^{\text{rd}}$, and 4^{th}, whence (using the above notation) $z_0 = 1, z_1 = 1$, $z_2 = 0$, $z_3 = 1$, $z_4 = 1$, and $z_5 = 0$. Hence, $X^\sharp = 110110$ and $X^\flat = 11011$. For this particular X, $X^* = 699093 + 390996 = 1090089 = 99 \times 11011 = 99X^\flat$. That this is no chance happening is shown in our first result.

Theorem 1: Let $X = x_n \dots x_0$ $(x_n > x_0)$ have truncated code $X^\flat = z_0 \dots z_{n-1}$. Then $X^* = 99X^\flat$.

Proof: Now

$$X = \sum_{i=0}^{n} x_i T^i \quad \text{and} \quad X' = \sum_{i=0}^{n} x_{n-i} T^i.$$

Suppose that $X^\sharp = z_0 \dots z_n$. Then the definitions of $z_0, \dots, z_n$ show that

$$X - X' = \sum_{i=0}^{n} (x_i + z_i T - x_{n-i} - z_{i-1}) T^i,$$

where we have written $z_{-1} = 0$. Hence,

$$\begin{aligned} X^* &= X - X' + (X - X')' \\ &= \sum_{i=0}^{n} (x_i + z_i T - x_{n-i} - z_{i-1} + x_{n-i} + z_{n-i} T - x_i - z_{n-i-1}) T^i \\ &= \sum_{i=0}^{n} (z_i T - z_{i-1\_} z_{n-i} T - z_{n-i-1}) T^i \\ &= \sum_{i=0}^{n-1} z_i T^{i+1} - \sum_{i=0}^{n-1} z_i T^{i+1} + T^2 \sum_{i=1}^{n} z_{n-i} T^{i-1} - \sum_{i=1}^{n} z_{n-i} T^{i-1} \\ &= (T^2 - 1) z_0 \dots z_{n-1} \\ &= 99 X^b. \ \square \end{aligned}$$

Theorem 1 shows that the number a_n we seek is the same as the number of different truncated codes X^b or, equivalently, codes $X^\sharp$ there are as X ranges over all $n+1$-digit numbers $X = x_0 \dots x_n$ $(x_n > x_0)$. The idea of a truncated code was introduced to allow Theorem 1 to be stated effectively, and from now on only the codes themselves will be considered. To help calculate a_n. we need to reformulate and formalize the definition of $X^\sharp$ given earlier. Define the *code* $X^\sharp$ of $X = x_n \dots x_0$ $(x_n > x_0)$ to be the number $y_n \dots y_0$, where the $y_n, \dots, y_0$ are defined inductively as follows. Let $y_n = 1$. For $i = 1, \dots, n$, define y_{n-i} to be 1 if *either* $x_{n-i} > x_i$ *or* $x_{n-i} = x_i$ and $y_{n-i+1} = 1$, and to be 0 otherwise, i.e., if *either* $x_{n-i} < x_i$ *or* $x_{n-i} = x_i$ and $y_{n-i+1} = 0$. This definition clearly accords with that given previously.

Theorem 2: The $n+1$-digit number $y_n \dots y_0$ is the code of some $n+1$-digit number $x_n \dots x_0$ $(x_n > x_0)$ if and only if: (i) each of $y_0, \dots, y_n$ is 0 or 1 and $y_0 = 0, y_n = 1$; (ii) if, for some $i = 0, \dots, n-1$, $y_{n-i} = 0$, and $y_{n-i-1} = 1$, then $y_{i+1} = 0$; (iii) if, for some $i = 0, \dots, n-1$, $y_{n-i} = 1$ and $y_{n-i-1} = 0$, then $y_{i+1} = 1$.

Proof: The *only if* part of the assertion follows directly from the definition of code just given. To establish the *if* part, suppose that $y_0, \dots, y_n$ satisfy conditions (i)-(iii). Let $w_n \dots w_0$ be the code of $y_n \dots y_0$. Then (ii) shows that $w_n = y_n = 1$. Either y_{n-1} is 0 or 1. Suppose first that $y_{n-1} = 0$. Then (ii) shows that $y_1 = 1$, whence $w_{n-1} = y_{n-1} = 0$. Suppose next that $y_{n-1} = 1$. Since y_1 is 0 or 1 and $w_n = 1$, the definition of w_{n-1} shows that $w_{n-1} = y_{n-1} = 1$. Therefore, in all cases, $w_{n-1} = y_{n-1}$. Continuing in this way, it can be shown that $w_{n-2} = y_{n-2}, \dots, w_0 = y_0$, whence $w_n \dots w_0 = y_n \dots y_0$, i.e.,$y_n \dots y_0$ is its own code. $\square$

3. THE CALCULATION OF a_n

We call any $n+1$-digit number $y_n \dots y_0$ satisfying conditions (i)-(iii) of Theorem 2 an $n+1$-digit code. Theorems 1 and 2 together show that a_n is simply the number of $n+1$-digit codes that there are, and it is this observation we use to calculate a_n. Trivially, the only two-digit code is 10 and the only three-digit code is 110. There are precisely three four-digit codes—1010, 1100, 1110—and three five-digit codes—10010, 11100, 11110. Thus, $a_1 = a_2 = 1$ and $a_3 = a_4 = 3$. It should be noted that, in each of the five-digit codes, the second and third digits are equal, and if the middle (i.e., third) digit is removed, then a four-digit code is obtained. Conversely, if each of the four-digit codes is extended by repeating its second digit, a five-digit code is obtained. These remarks explain why $a_4 = a_3$. We now extend these ideas.

Suppose that $y_{2n} \dots y_{n+1} y_n y_{n-1} \dots y_0$ is a $2n+1$-digit code $(n \geq 1)$. Then conditions (ii) and (iii) of Theorem 2 show that $y_{n+1} = y_n$. It follows easily that $y_{2n} \dots y_{n+1} y_{n-1} \dots y_0$ is a $2n$-digit code. Conversely, if $z_{2n-1} \dots z_n z_{n-1} \dots z_0$ is a $2n$-digit code, then $z_{2n-1} \dots z_n z_n z_{n-1} \dots z_0$ is a $2n+1$-digit code. Hence, there is a bijection between the set of $2n+1$-digit codes and the set of $2n$-digit codes, whence $a_{2n} = a_{2n-1}$.

To help find a recurrence relation satisfied by the a_n, we consider, for each natural number n, the set $\mathcal{S}_n$ comprising all $n+1$-digit numbers $s_n \dots s_0$ satisfying: (a) each of $s_0, \dots, s_n$ is 0 or 1; (b) if, for some $i = 0, \dots, n-1$, $s_{n-i} = 0$ and $s_{n-i-1} = 1$, then $s_{i+1} = 0$; (c) if, for some $i = 0, \dots, n-1$, $s_{n-i} = 1$ and $s_{n-i-1} = 0$, then $s_{i+1} = 1$. Thus, a_n is the number of those elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = 1$ and $s_0 = 0$. If an element of $\mathcal{S}_n$ is taken, and each 0 in it is changed to 1, and each 1 to 0, then another element of $\mathcal{S}_n$ is obtained. Hence, the number of elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = 0$ and $s_0 = 1$ is also a_n. Similarly, the number of those elements $s_n \dots s_0$ in $\mathcal{S}_n$ for which $s_n = s_0 = 0$ is the same number as those for which $s_n = s_0 = 1$; we denote this common number by b_n.

The members $s_n \dots s_0$ of $\mathcal{S}_n$ $(n \geq 3)$ for which $s_n = s_0 = 0$, other than the one comprising all zeros, have one of the forms,

$$0 \dots 0X0 \dots 0,$$

in which there are r initial zeros, r final zeros, and X is an $n-2r+1$-digit code, for some natural number r satisfying $2r \leq n-1$. Conversely, each $n+1$-digit number of the above form lies in $\mathcal{S}_n$ and has both its initial and final digits zero. Thus, for $n \geq 3$,

$$b_n = \begin{cases} a_{n-2} + \cdots + a_4 + a_2 + 1 & (n \text{ even}) \\ a_{n-2} + \cdots + a_3 + a_1 + 1 & (n \text{ odd}). \end{cases}$$

Trivially, $b_2 = b_1 = 1$.

Since $a_{2n} = a_{2n-1}$, we need only calculate a_{2n-1}. To this end, we note that every $2n+2$-digit code has one of the forms,

$$1X0,\ 1Y0,\ 1Z0,$$

where $X, Y, Z \in \mathcal{S}_{2n-1}$ are such that $X = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = 1, s_0 = 0$, $Y = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = 0, s_0 = 1$, and $Z = s_{2n-1} \dots s_0$ satisfies $s_{2n-1} = s_0 = 1$. Conversely, each such X, Y, Z gives rise, respectively, to a $2n+2$-digit code $1X0, 1Y0, 1Z0$. In view of our earlier remarks, the number of possible Xs is a_{2n-1}, the number of possible Ys is a_{2n-1}, and the number of possible Zs is b_{2n-1}. Hence, for $n \geq 2$,

$$\begin{aligned} a_{2n+1} &= a_{2n-1} + a_{2n-1} + b_{2n-1} \\ &= a_{2n-1} + a_{2n-1} + a_{2n-3} + \cdots a_3 + a_1 + 1 \\ &= 2a_{2n-1} + a_{2n-3} + \cdots + a_3 + a_1 + 1. \end{aligned}$$

This recurrence relation enables us to prove our main result.

Theorem 3: For each natural number n, $a_{2n} = a_{2n-1} = F_{2n}$, i.e., $a_n = F_{2[\frac{n+1}{2}]}$.

Proof: Since $a_{2n} = a_{2n-1}$, it remains only to prove that $a_{2n-1} = F_{2n}$. We do this by induction on n. The cases $a_1 = 1 = F_2$ and $a_3 = 3 = F_4$ have been established earlier. Suppose that $a_{2k-1} = F_{2k}$, for $k = 1, \ldots, n$, where $n \geq 2$. Then

$$\begin{aligned} a_{2n+1} &= 2a_{2n-1} + a_{2n-3} + \cdots + a_3 + a_1 + 1 \\ &= 2F_{2n} + F_{2n-2} + \cdots + F_4 + F_2 + 1 \\ &= F_{2n} + (F_{2n+1} - F_{2n-1}) + (F_{2n-1} - F_{2n-3}) + \cdots + (F_5 - F_3) + (F_3 - F_1) + 1 \\ &= F_{2n} + F_{2n+1} \\ &= F_{2n+2}. \end{aligned}$$

This completes the proof by induction. □

An easy exercise shows that, for $n \geq 2$,

$$b_{2n} = b_{2n-1} = F_{2n-2} + \cdots + F_4 + F_2 + 1 = F_{2n-1}.$$

Since $b_2 = b_1 = F_1$, the b_ns are the Fibonacci numbers with odd suffixes, in the same way that the a_ns are those with even suffixes.

4. CONCLUDING REMARKS

Our original problem extends in the obvious way to include as initial numbers every $n+1$-digit number X whose reverse X' satisfies $X' \leq X$. In this wider context, we ask: How many end numbers are now possible and what are they? The *extra* initial numbers that have to be considered either generate the end number 0 or have the form YXY', where Y is an r-digit number, X is an $n-2r+1$-digit number whose initial digit exceeds its final one, and the natural number r satisfies $2r \leq n-1$. This latter form gives rise to the a_{n-2r+1} end numbers $99(10^{r-1})$ code of X. Thus, the total number of end numbers now possible is:

$$\begin{cases} a_n + a_{n-2} + \cdots + a_2 + 1 = F_n + F_{n-2} + \cdots + F_2 + 1 = F_{n+1} & (n \text{ even}), \\ a_n + a_{n-2} + \cdots + a_1 + 1 = F_{n+1} + F_{n-1} + \cdots F_2 + 1 = F_{n+2} & (n \text{ odd}). \end{cases}$$

Denoting this latter number by α_n, we see that $\alpha_{2n} = \alpha_{2n-1} = F_{2n+1}$, i.e., $\alpha_n = F_{2[\frac{n+1}{2}]+1}$.

Although our discussion has been concerned exclusively with base 10 arithmetic, it generalizes, with only minor modifications, to an arbitrary base m. The main change required is that in Theorem 1 the number 99 has to be replaced by $m^2 - 1$. A propos the concluding remarks of the opening paragraph, the unique end number generated by a three-digit number abc $(a > c)$ to base m is the four-digit number $1\,0\,m-2\,m-1$, which equals $(m-1)(m+1)^2$ and is a square precisely when $m-1$ is; this is fortuitously so when $m=10$. On the other hand, the reverse of $1\,0\,m-2\,m-1$ is $m-1\,m-2\,0\,1$, which equals $(m^2-1)^2$ and is always square.

REFERENCE

1. W. W. Rouse Ball. *Mathematical Recreations and Essays.* London: Macmillan, 1939.

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Announcement

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