A NOTE ON CHOUDHRY'S RESULTS

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1. INTRODUCTION

In [4] Fell, Graz, and Paasche proved that if the equation

$$x^n + y^n = z^n, \tag{1}$$

where $n \geq 2$ is an integer, has a solution in positive integers $x < y < z$, then

$$x^2 > 2y + 1. \tag{2}$$

In 1969 M. Perisastri (see [7], p. 226) proved that

$$x^2 > z. \tag{3}$$

In [1] it was proved that

$$x^2 > 2z + 1. \tag{4}$$

A. Choudhry (in [3]) improved the inequality (4) to the form

$$x^{n/(n-1)} > z. \tag{5}$$

In fact, from the proof given by Choudhry [3], it follows that

$$z < C(n)x^{n/(n-1)}, \tag{6}$$

where

$$C(n) = 2^{1/n} / n^{1/(n-1)}, \quad n > 1. \tag{7}$$

In [2] we improved the constant (7) to the form

$$C_1(n) = 2^{1/2n} / n^{1/(n-1)} < C(n). \tag{8}$$

In this note, we shall prove the following

Theorem: Let $C(j,k;n) = j^{1/n} / k^{1/(n-1)}$ and let equation (1) have a solution in positive integers $x < y < z$, then

$$z < \begin{cases} C(2, n; n)x^{n/(n-1)}, & \text{if } z - y = 1, \\ C(\sqrt{2}, 2n; n)x^{n/(n-1)}, & \text{if } z - y = 2, \\ C(\sqrt{2}, 2^n; n)x^{n/(n-1)}, & \text{if } z - y > 2. \end{cases}$$

Proof of the Theorem: Suppose equation (1) has a solution in positive integers $x < y < z$. Then we have

$$x^n = z^n - y^n = (z - y)(z^{n-1} + z^{n-2}y + \cdots + y^{n-1}). \tag{9}$$

We note that

$$z^{n-1} + z^{n-2}y + \cdots + y^{n-1} > n(zy)^{(n-1)/2}. \tag{10}$$

On the other hand, if $x < y < z$, we have, by (1),

$$y > (1/2)^{1/n} z. \tag{11}$$

From (10) and (11), we obtain

$$z^{n-1} + z^{n-2}y + \cdots + y^{n-1} > (n/2^{(n-1)/2n})z^{n-1}. \tag{12}$$

It is well-known (see [7], Ch. 11) that if n, x, y, z are positive integers with $x < y < z$ and $(x, y, z) = 1$ such that (1) holds, then there exist $\delta \in \{0, 1\}$ and positive integers a, d with $d|n$ such that

$$z - y = 2^{\delta} d^{-1} a^n. \tag{13}$$

From (13), it follows that if $z - y > 2$ then

$$z - y \geq 2^n / n \quad (cf.[5]). \tag{14}$$

From (12) and (9), we obtain

$$x^n > (z-y)(n/2^{(n-1)/2n})z^{n-1}. \tag{15}$$

From (15) and (14), we have

$$x^n > C_2(n)/2^{(n-1)/2n} z^{n-1}, \tag{16}$$

where

$$C_2(n) = \begin{cases} n, & \text{if } z - y = 1, \\ 2n, & \text{if } z - y = 2, \\ 2, & \text{if } z - y > 2. \end{cases}$$

Now, by (16), the Theorem follows. □

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