

# FORMULAS FOR $a + a^2 2^p + a^3 3^p + \dots + a^n n^p$

G. F. C. de Bruyn

Department of Mathematics, University of Stellenbosch, Stellenbosch, South Africa

(Submitted June 1993)

## 1. INTRODUCTION

Let  $S_{a,p}(n) = a + a^2 2^p + a^3 3^p + \dots + a^n n^p$ , with  $n \in N$ ,  $p \in N$ , and  $a \in R$  ( $a \neq 0$ ,  $a \neq 1$ ), where  $N$  and  $R$  are, respectively, the sets of positive integers and real numbers.

In [2] N. Gauthier used a calculus-based method to evaluate  $S_{a,p}(n)$ . He wrote  $S_{a,p}(n)$  as  $a^n$  times a polynomial of degree  $p$  in  $n$  plus a term which is  $n$ -independent. The coefficients are then determined recursively.

In this paper methods similar to those used in [1] are employed to derive various formulas for  $S_{a,p}(n)$ . Recurrence formulas in terms of powers of  $n$  and of  $n+1$  are given. Explicit expressions for  $S_{a,p}(n)$  in determinant form in terms of  $n$  and of  $n+1$  are then derived from these formulas. These determinants are finally used to write  $S_{a,p}(n)$  in terms of polynomials of degree  $p$  in  $n$  and in  $n+1$ .

## 2. FORMULAS IN TERMS OF POWERS OF $n+1$

### 2.1 A Recurrence Formula

Let  $n \in N$ . For  $k \in N$  and  $a \in R$  ( $a \neq 0$ ,  $a \neq 1$ ), let

$$S_{a,k}(n) = a + a^2 2^k + a^3 3^k + \dots + a^n n^k = \sum_{r=0}^n a^r r^k$$

and take

$$S_{a,0}(n) = 1 + a + a^2 + \dots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$

Then

$$\begin{aligned} a^{n+1}(n+1)^k &= S_{a,k}(n+1) - S_{a,k}(n) \\ &= \sum_{r=0}^n a^{r+1}(r+1)^k - S_{a,k}(n) \\ &= \sum_{r=0}^n \left( a^{r+1} \sum_{i=0}^k \binom{k}{i} r^i \right) - S_{a,k}(n) \\ &= a \sum_{i=0}^k \left( \binom{k}{i} \sum_{r=0}^n a^r r^i \right) - S_{a,k}(n) \\ &= a \sum_{i=0}^k \binom{k}{i} S_{a,i}(n) - S_{a,k}(n). \end{aligned} \tag{2.1.1}$$

The equation

$$a(S+1)^k - S^k = a^{n+1}(n+1)^k, \tag{2.1.2}$$

in which the binomial power on the left-hand side is expanded and  $S^i$  ( $i = 0, 1, 2, \dots, k$ ) are then replaced by  $S_{a,i}(n)$ , provides a mnemonic for (2.1.1).

For example, for  $k = 1$ , formula (2.1.2) gives

$$a(S+1) - S = a^{n+1}(n+1),$$

and so

$$(a-1)S_{a,1}(n) + a \binom{a^{n+1}-1}{a-1} = a^{n+1}(n+1).$$

Hence,

$$S_{a,1}(n) = \frac{a^{n+1}}{a-1}(n+1) - \frac{a}{(a-1)^2}(a^{n+1}-1). \quad (2.1.3)$$

Also, by (2.1.2), with  $k = 2$ ,

$$a^{n+1}(n+1)^2 = a(S+1)^2 - S^2 = a(S^2 + 2S+1) - S^2,$$

which implies that

$$(a-1)S_{a,2}(n) + 2aS_{a,1}(n) + a \binom{a^{n+1}-1}{a-1} = a^{n+1}(n+1)^2.$$

Thus, by (2.1.3),

$$S_{a,2}(n) = \frac{a^{n+1}}{a-1}(n+1)^2 - \frac{2a^{n+2}}{(a-1)^2}(n+1) + \frac{a^{n+1}-1}{(a-1)^3}(a^2+a).$$

## 2.2 $S_{a,p}(n)$ as a Determinant

Let  $p \in \mathbb{N}$  and let  $k = 1, 2, \dots, p$  in (2.1.1). It follows, applying Cramer's rule to these  $p$  equations together with the equation  $S_{a,0}(n) = \frac{a^{n+1}-1}{a-1}$ , that

$$S_{a,p}(n) = \frac{a^{n+p}}{(a-1)^p} \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a^{-n} \binom{a^{n+1}-1}{a-1} \\ 1 & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 & n+1 \\ 1 & \binom{2}{1} & \frac{a-1}{a} & 0 & \dots & 0 & 0 & (n+1)^2 \\ \vdots & \vdots \\ 1 & \binom{p-1}{1} & \binom{p-1}{2} & \dots & \binom{p-1}{p-2} & \frac{a-1}{a} & (n+1)^{p-1} \\ 1 & \binom{p}{1} & \binom{p}{2} & \dots & \binom{p}{p-2} & \binom{p}{p-1} & (n+1)^p \end{vmatrix} \quad (2.2.1)$$

$$= \frac{a^{n+p}}{(a-1)^p} p! \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a^{-n} \binom{a^{n+1}-1}{a-1} \\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 & \frac{n+1}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \dots & 0 & 0 & \frac{(n+1)^2}{2!} \\ \vdots & \vdots \\ \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & \frac{1}{(p-3)!} & \dots & \frac{1}{1!} & \frac{a-1}{a} & \frac{(n+1)^{p-1}}{(p-1)!} \\ \frac{1}{p!} & \frac{1}{(p-1)!} & \frac{1}{(p-2)!} & \dots & \frac{1}{2!} & \frac{1}{1!} & \frac{(n+1)^p}{p!} \end{vmatrix} \quad (2.2.2)$$

### 2.3 $S_{a,p}(n)$ in Terms of a Polynomial

By expanding the determinant (2.2.2) with respect to the last column,

$$S_{a,p}(n) = \sum_{r=0}^{p-1} \alpha_r (n+1)^{p-r} + \alpha_p a^{-(n+1)} (a^{n+1} - 1), \quad (2.3.1)$$

with  $\alpha_0 = \frac{a^{n+1}}{a-1}$  and, for  $r = 1, 2, \dots, p$ ,

$$\alpha_r = \binom{p}{r} \frac{a^{n+1}}{a-1} \frac{a^r}{(a-1)^r} r! (-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots & \dots & \dots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \dots & \dots & \dots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}.$$

Now, let  $f_0(a) = 1$  and, for  $r = 1, 2, 3, \dots$ ,

$$f_r(a) = \frac{a^r}{(a-1)^r} r! (-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots & \dots & \dots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \dots & \dots & \dots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}. \quad (2.3.2)$$

Then, by (2.3.1),

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1} \sum_{r=0}^{p-1} \binom{p}{r} f_r(a) (n+1)^{p-r} + f_p(a) \left( \frac{a^{n+1} - 1}{a-1} \right). \quad (2.3.3)$$

The real numbers  $f_r(a)$ ,  $r = 1, 2, 3, \dots$ , can also be calculated recursively in the following way. Consider, for  $r \in \mathbb{N}$ ,

$$\begin{aligned} a^n f_r(a) &= \frac{a^{n+r}}{(a-1)^r} r! (-1)^r \begin{vmatrix} \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots & \dots & \dots & \frac{1}{1!} & \frac{a-1}{a} \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \dots & \dots & \dots & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} \\ &= \frac{a^{n+r}}{(a-1)^r} r! \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ \frac{1}{1!} & \frac{a-1}{a} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{a-1}{a} & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \dots & \dots & \dots & \frac{1}{1!} & \frac{a-1}{a} & 0 \\ \frac{1}{r!} & \frac{1}{(r-1)!} & \dots & \dots & \dots & \frac{1}{2!} & \frac{1}{1!} & 0 \end{vmatrix} \end{aligned}$$

Observe that the last determinant differs from that of  $S_{a,r}(n)$ , as obtained by setting  $p = r$  in (2.2.2), only with respect to the last column. It follows [cf. (2.1.1)] that  $f_0(a), f_1(a), f_2(a), \dots$  satisfy the recurrence formula

$$f_0(a) = 1, a \sum_{i=0}^r \binom{r}{i} f_i(a) - f_r(a) = 0 \quad (r = 1, 2, 3, \dots). \quad (2.3.4)$$

Here the equation

$$a(f+1)^r - f^r = 0, \quad (2.3.5)$$

in which the binomial power is expanded and  $f^r$  ( $r = 0, 1, 2, 3, \dots$ ) are then replaced by  $f_r(a)$ , provides a mnemonic for (2.3.4).

Note that (2.3.4), with  $a = 1$ , is the well-known recurrence formula for the Bernoulli numbers. The real numbers  $f_r(a)$ ,  $r = 0, 1, 2, 3, \dots$ , could therefore be called the  $a$ -Bernoulli numbers. For example, by (2.3.2) or, recursively, by (2.3.5),

$$f_0(a) = 1, f_1(a) = \frac{-a}{a-1}, f_2(a) = \frac{a+a^2}{(a-1)^2}, \text{ and } f_3(a) = \frac{-(a+4a^2+a^3)}{(a-1)^3}. \quad (2.3.6)$$

Hence, 1, -2, 6, -26 are the first four 2-Bernoulli numbers.

### 3. FORMULAS IN TERMS OF POWERS OF $n$

Let  $n \in N$ . For  $k \in N$  and  $a \in R$  ( $a \neq 0, a \neq 1$ ), let

$$S_{a,k}(n) = a + a^2 2^k + a^3 3^k + \dots + a^n n^k = \sum_{r=1}^n a^r r^k, \quad S_{a,k}(0) = 0,$$

and take

$$S_{a,0}(n) = a + a^2 + \dots + a^n = \frac{a^{n+1} - a}{a - 1}.$$

Then, arguing as in Section 2.1,

$$a^n n^k = S_{a,k}(n) - S_{a,k}(n-1) = S_{a,k}(n) - \frac{1}{a} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} S_{a,i}(n).$$

Hence,

$$a^{n+1} n^k = a S_{a,k}(n) - \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} S_{a,i}(n). \quad (3.1)$$

The equation

$$aS^k - (S-1)^k = a^{n+1} n^k,$$

in which the binomial power on the left-hand side is expanded and  $S^i$  ( $i = 0, 1, 2, \dots, k$ ) are then replaced by  $S_{a,i}(n)$ , provides a mnemonic for (3.1).

Furthermore, methods similar to those employed in Sections 2.2 and 2.3 can be used to derive the following results from (3.1).

$$S_{a,p}(n) = \frac{a^{n+1}}{(a-1)^p} p! \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a^{-(n+1)} \left( \frac{a^{n+1}-a}{a-1} \right) \\ \frac{1}{1!} & a-1 & 0 & 0 & \dots & 0 & 0 & \frac{n}{1!} \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \dots & 0 & 0 & \frac{n^2}{2!} \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \dots & 0 & 0 & \frac{n^3}{3!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{(-1)^p}{(p-1)!} & \frac{(-1)^{p+1}}{(p-2)!} & \dots & \dots & \dots & \frac{1}{1!} & a-1 & \frac{n^{p-1}}{(p-1)!} \\ \frac{(-1)^{p+1}}{p!} & \frac{(-1)^{p+2}}{(p-1)!} & \dots & \dots & \dots & -\frac{1}{2!} & \frac{1}{1!} & \frac{n^p}{p!} \end{vmatrix},$$

and

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1} \sum_{r=0}^{p-1} \binom{p}{r} g_r(a) n^{p-r} + g_p(a) \left( \frac{a^{n+1}-a}{a-1} \right), \quad (3.2)$$

with  $g_0(a) = 1$  and, for  $r = 1, 2, 3, \dots$ ,

$$g_r(a) = \frac{r!(-1)^r}{(a-1)^r} \begin{vmatrix} \frac{1}{1!} & a-1 & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{2!} & \frac{1}{1!} & a-1 & 0 & \dots & 0 & 0 \\ \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} & a-1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^r}{(r-1)!} & \frac{(-1)^{r+1}}{(r-2)!} & \dots & \dots & \dots & \frac{1}{1!} & a-1 \\ \frac{(-1)^{r+1}}{r!} & \frac{(-1)^{r+2}}{(r-1)!} & \dots & \dots & \dots & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}.$$

The real numbers  $g_r(a)$ ,  $r = 1, 2, 3, \dots$ , can also be calculated recursively in a similar way as it is done in the case of  $f_r(a)$ ,  $r = 1, 2, 3, \dots$ , in Section 2.3. However, it is easier to observe that, by (2.3.3) and (3.2) (comparing  $n$ -free terms),  $f_r(a) = ag_r(a)$  for each  $r \in N$ . Hence, by (3.2),

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1} n^p + \frac{a^n}{a-1} \sum_{r=1}^{p-1} \binom{p}{r} f_r(a) n^{p-r} + f_p(a) \left( \frac{a^n-1}{a-1} \right), \quad \text{for } p > 1. \quad (3.3)$$

For example, let  $p = 2$  in (3.3). Then, by (2.3.6),

$$\begin{aligned} S_{a,2}(n) &= \frac{a^{n+1}}{a-1} n^2 + \frac{2a^n}{a-1} f_1(a)n + f_2(a) \left( \frac{a^n-1}{a-1} \right) \\ &= \frac{a^{n+1}}{a-1} n^2 - \frac{2a^{n+1}}{(a-1)^2} n + \frac{(a+a^2)(a^n-1)}{(a-1)^3}. \end{aligned}$$

In particular,

$$S_{3,2}(n) = \sum_{r=1}^n 3^r r^2 = \frac{3^{n+1}}{2} n^2 - \frac{3^{n+1}}{2} n + \frac{3^{n+1}}{2} - \frac{3}{2}.$$

**REFERENCES**

1. G. F. C. de Bruyn & J. M. de Villiers. "Formulas for  $+2^p + 3^p + \dots + n^p$ ." *The Fibonacci Quarterly* **32.3** (1994):271-76.
2. N. Gauthier. "Derivation of a Formula for  $\sum r^k x^r$ ." *The Fibonacci Quarterly* **27.5** (1989): 402-08.
3. D. Kalman. "Sums of Powers by Matrix Methods." *The Fibonacci Quarterly* **28.1** (1990): 60-71.

AMS Classification Numbers: 05A10, 05A19, 11B68



**GENERALIZED PASCAL TRIANGLES AND PYRAMIDS:  
THEIR FRACTALS, GRAPHS, AND APPLICATIONS**

by Dr. Boris A. Bondarenko

*Associate member of the Academy of Sciences of the Republic of Uzbekistan, Tashkent*

Translated by Professor Richard C. Bollinger

*Penn State at Erie, The Behrend College*

This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, see *The Fibonacci Quarterly* 31.1 (1993):52.

The translation of the book is being reproduced and sold with the permission of the author, the translator, and the "FAN" Edition of the Academy of Science of the Republic of Uzbekistan. The book, which contains approximately 250 pages, is a paperback with a plastic spiral binding. The price of the book is \$31.00 plus postage and handling where postage and handling will be \$6.00 if mailed anywhere in the United States or Canada, \$9.00 by surface mail or \$16,00 by airmail elsewhere. A copy of the book can be purchased by sending a check make out to THE FIBONACCI ASSOCIATION for the appropriate amount along with a letter requesting a copy of the book to: MR. RICHARD S. VINE, SUBSCRIPTION MANAGER, THE FIBONACCI ASSOCIATION, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053.