ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-784 ***Proposed by Herta Freitag, Roanoke, VA***

Show that for all n, $\alpha^{n-1}\sqrt{5} - L_{n-1}/\alpha$ is a Lucas number.

B-785 ***Proposed by Jane E. Friedman, University of San Diego, San Diego, CA***

Let $a_0 = a_1 = 1$ and let $a_n = 5a_{n-1} - a_{n-2}$ for $n \geq 2$. Prove that $a_{n+1}^2 + a_n^2 + 3$ is a multiple of $a_n a_{n+1}$ for all $n \geq 1$.

B-786 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat State, India***

If $F_{n+2k}^2 = aF_{n+2}^2 + bF_n^2 + c(-1)^n$, where a, b, and c depend only on k but not on n, find a, b, and c.

B-787 ***Proposed by H.-J. Seiffert, Berlin, Germany***

For $n \geq 0$ and $k > 0$, it is known that F_{kn}/F_k and P_{kn}/P_k are integers. Show that these two integers are congruent modulo $R_k - L_k$.

[**Note:** P_n and $R_n = 2Q_n$ are the Pell and Pell-Lucas numbers, respectively, defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 1$, $Q_1 = 1$.]

B-788 ***Proposed by Russell Jay Hendel, University of Louisville, Louisville, KY***

(a) Let $G_n = F_{n^2}$. Prove that $G_{n+1} \sim L_{2n+1}G_n$.

[**Note:** $f(n) \sim g(n)$ means that *f* is asymptotic to *g*, that is, $\lim_{n\to\infty} f(n)/g(n) = 1$.]

(b) Find the error term. More specifically, find a constant C such that $G_{n+1} \sim L_{2n+1}G_n + CG_{n-1}$.

B-789 ***Proposed by Richard André-Jeannin, Longwy, France***

The Lucas polynomials, $L_n(x)$, are defined by $L_0 = 2,\ L_1 = x$, and $L_n = xL_{n-1} + L_{n-2}$, for $n \geq 2$.

Find a differential equation satisfied by $L_n^{(k)}$, the k^{th} derivative of $L_n(x)$, where k is a nonnegative integer.

SOLUTIONS

Inequality for All

B-752 ***Proposed by Richard André-Jeannin, Longwy, France***
(Vol. 31, no. 4, November 1993)

Consider the sequences $\langle U_n\rangle$ and $\langle V_n\rangle$ defined by the recurrences $U_n = PU_{n-1} - QU_{n-2}$, $n \geq 2$, with $U_0 = 0, U_1 = 1$, and $V_n = PV_{n-1} - QV_{n-2}$, $n \geq 2$, with $V_0 = 2, V_1 = P$, where P and Q are real numbers with $P > 0$ and $\Delta = P^2 - 4Q > 0$. Show that, for $n \geq 0, U_{n+1} \geq (P/2)U_n$ and $V_{n+1} \geq (P/2)V_n$.

Solution by A. N. 't Woord, Eindhoven Univ. of Tech., Eindhoven, The Netherlands

Let $\langle W_n\rangle$ be any sequence that satisfies $W_n = PW_{n-1} - QW_{n-2}$ for $n \geq 2$ and $W_1 \geq (P/2)W_0 \geq 0$. Using induction on *n*, we shall show that $W_{n+1} \geq (P/2)W_n \geq 0$. We already know this for $n = 0$, so suppose the inequality holds for $n-1$. Then

$$\begin{aligned} W_{n+1} &= PW_n - QW_{n-1} \geq PW_n - (2Q/P)W_n \\ &= (P/2)W_n + (P/2 - 2Q/P)W_n \\ &= (P/2)W_n + (\Delta/2P)W_n \geq (P/2)W_n \geq 0. \end{aligned}$$

This gives the required result for both the sequences $\langle U_n\rangle$ and $\langle V_n\rangle$.

Note that the same style proof shows that strict inequality holds for $n > 0$.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, Hans Kappus, H.-J. Seiffert, Lawrence Somer, J. Suck, and the proposer.

An Old Determinant

B-753 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. of Sci., Gujarat State, India***
(Vol. 31, no 4, November 1993)

Prove that, for all positive integers *n*,

$$\begin{vmatrix} F_{n-1}^3 & F_n^3 & F_{n+1}^3 & F_{n+2}^3 \\ F_n^3 & F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 \\ F_{n+1}^3 & F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 \\ F_{n+2}^3 & F_{n+3}^3 & F_{n+4}^3 & F_{n+5}^3 \end{vmatrix} = 36.$$

Comment by J. Suck, Essen, Germany

"Surely you must be joking! . . . The world's leading previously published problems surveyor . . . was taken in?" This problem is the same as problem H-25 which was proposed by Erbacher and Fuchs in *The Fibonacci Quarterly* in 1964.

The editor apologizes for repeating this problem. Many readers pointed out this duplication. See the simple solution by C. R. Wall that was originally printed in this Quarterly **2.3** *(Oct. 1964):207.*

Generalization by H.-J. Seiffert, Berlin, Germany

For the positive integer p, let

$$A_n(p) = \begin{vmatrix} F_{n-1}^{p-1} & F_n^{p-1} & \cdots & F_{n+p-2}^{p-1} \\ F_n^{p-1} & F_{n+1}^{p-1} & \cdots & F_{n+p-1}^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+p-2}^{p-1} & F_{n+p-1}^{p-1} & \cdots & F_{n+2p-3}^{p-1} \end{vmatrix}_{p\times p},$$

where n is an arbitrary integer. According to a result of D. Jarden (see [1], p. 85, exercise 30), we have

$$\sum_{k=0}^{p} \begin{bmatrix} p \\ k \end{bmatrix} (-1)^{\lceil (p-k)/2 \rceil} F_{N+k}^{p-1} = 0 \tag{1}$$

for all integers N, where $\left[\begin{smallmatrix} p \\ k \end{smallmatrix}\right]$ is the Fibonomial coefficient defined by

$$\begin{bmatrix} p \\ k \end{bmatrix} = \frac{F_p F_{p-1} \cdots F_{p-k+1}}{F_k F_{k-1} \cdots F_1}$$

with $\left[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} p \\ p \end{smallmatrix}\right] = 1$. Here $\lceil x \rceil$ denotes the least integer greater than or equal to x. Letting $N = n - 2 + j$ in equation (1) gives

$$F_{n+p-2+j}^{p-1} = -(-1)^{\lceil p/2 \rceil} F_{n-2+j}^{p-1} - \sum_{k=1}^{p-1} \begin{bmatrix} p \\ k \end{bmatrix} (-1)^{\lceil (p-k)/2 \rceil} F_{n+k-2+j}^{p-1}$$

for all $j = 0, 1, 2, \ldots, p-1$. Thus, we have $A_n(p) = (-1)^{p+\lceil p/2 \rceil} A_{n-1}(p)$ for all integers n, which implies that

$$A_n(p) = (-1)^{(n-m)(p+\lceil p/2 \rceil)} A_m(p)$$

for all integers m and n. Letting $m = 2 - p$ allows us to calculate $A_n(p)$. The results are given in the following table.

p	1	2	3	4	5	6
$A_n(p)$	1	$(-1)^n$	$2(-1)^n$	36	13824	$324000000(-1)^n$

Reference

1. Donald E. Knuth. *The Art of Computer Programming.* Vol. 1. Reading, Mass.: Addison-Wesley, 1973.

Dresel found that if the Fibonacci numbers are replaced by Lucas numbers in the original proposal, then the value of the determinant obtained is 562,500. He also showed that if the original determinant is enlarged to be r × r for r > 4, then the value of the determinant is 0.

This follows from the identity $F_{n+2}^3-3F_{n+1}^3-6F_n^3+3F_{n-1}^3+F_{n-2}^3=0$, *which shows that the rows are linearly dependent.*

Also solved by Marjorie Bicknell-Johnson, Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Samih A. Obaid, H.-J. Seiffert, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

A Summing of Pells

B-754 ***Proposed by Joseph J. Kostal, University of Illinois at Chicago, IL (Vol. 32, no. 1, February 1994)***

Find closed form expressions for

$$\sum_{k=1}^{n} P_k \quad \text{and} \quad \sum_{k=1}^{n} Q_k.$$

The Pell numbers P_n and their associated numbers Q_n are defined by

$$P_{n+2}=2P_{n+1}+P_n, \quad P_0=0, \ P_1=1;$$
$$Q_{n+2}=2Q_{n+1}+Q_n, \ Q_0=1, \ Q_1=1.$$

Solution by Glenn A. Bookhout, Durham, NC and by H.-J. Seiffert, Berlin, Germany (independently)

Let $\langle G_n \rangle$ be any sequence that satisfies the recurrence $G_{n+2}=2G_{n+1}+G_n$. Then

$$\sum_{k=1}^{n} G_{k+2}=2\sum_{k=1}^{n} G_{k+1}+\sum_{k=1}^{n} G_k.$$

Thus,

$$\sum_{k=1}^{n} G_k+G_{n+1}+G_{n+2}-G_1-G_2=2\sum_{k=1}^{n} G_k+2G_{n+1}-2G_1+\sum_{k=1}^{n} G_k.$$

Hence,

$$\sum_{k=1}^{n} G_k=\frac{1}{2}(G_{n+2}-G_{n+1}-G_2+G_1)=\frac{1}{2}(G_{n+1}+G_n-G_2+G_1).$$

Several solvers pointed out that this and similar problems can be solved using the Binet forms and the formula for the sum of a finite geometric progression: $\sum_{k=1}^{n} x^k=(x-x^{n+1})/(1-x)$. *Some of the other equivalent answers obtained were:* $\sum P_k=(P_{n+1}+P_n-1)/2=(Q_{n+1}-1)/2$ *and* $\sum Q_k=(Q_{n+1}+Q_n)/2-1=P_{n+1}-1$. *Haukkanen points out that Horadam showed in this Quarterly* **3.2** *(1965):161-77 that, if the sequence* w_n *is defined by* $w_{n+2}=cw_{n+1}-dw_n$ *for* $n\geq 0$ *with* $w_0=a$ *and* $w_1=b$ *and* $c\neq d+1$, *then*

$$\sum_{k=0}^{n} w_k=\frac{w_{n+2}-b-(c-1)(w_{n+1}-a)}{c-d-1}.$$

Gauthier found that for any integers s and t,

$$\sum_{k=1}^{n} x^n G_{sk+t}=\frac{(-1)^s x^{n+1} G_{sn+t}-x^n G_{s(n+1)+t}+G_{s+t}+(-1)^{s+1} x G_t}{1-2x(G_s+G_{s-1})+(-1)^s x^2}.$$

Also solved by Seung-Jin Bang, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, Russell Euler, Herta T. Freitag, N. Gauthier, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Hans Kappus, H. K. Krishnapriyan, Carl Libis, Bob Prielipp, Sahib Singh, David C. Terr, and the proposer.

An Interleaving of Pells

B-755 *Proposed by Russell Jay Hendel, Morris College, Sumter, SC (Vol. 32, no. 1, February 1994)*

Find all nonnegative integers m and n such that $P_n = Q_m$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

The only solutions are $(m, n) = (0, 1)$ or $(1, 1)$. First, n cannot be 0, since Q_m is never 0. Next, for $n = 1$, we obtain the two solutions listed above. Finally, for $n > 1$, it is straightforward to show that $Q_n = P_{n-1} + P_n = P_{n+1} - P_n$. Since $\langle P_n \rangle$ is a strictly increasing sequence of positive integers for $n > 1$, this yields $P_n < Q_n < P_{n+1}$, so Q_m cannot equal P_n for any $n > 1$.

Also solved by Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, C. Georghiou, Hans Kappus, Murray S. Klamkin, Wayne L. McDaniel, H.-J. Seiffert, Sahib Singh, David C. Terr, and the proposer.

A Fibonacci Formula for P_n

B-756 *Proposed by the editor (Vol. 32, no. 1, February 1994)*

Find a formula expressing the Pell number P_n in terms of Fibonacci and/or Lucas numbers.

Editorial Note: *Although some very ingenious solutions were submitted, none had the elegance that might be expected of our distinguished panel of solvers. This problem will thus be kept open for another six months.*

Fibonacci-Pell Congruences

B-757 *Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 32, no. 1, February 1994)*

Show that for $n > 0$,

(a) $$P_{3n-1} \equiv F_{n+2} \pmod{13},$$

(b) $$P_{3n+1} \equiv (-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod{7}.$$

Solution by Bill Correll, Jr., Student, Denison University, Granville, OH

(b) Modulo 7, the Pell numbers P_n repeat in the sequence 0, 1, 2, 5, 5, 1, ... Thus, P_{3n+1} repeats in the sequence 1, 5, 1, 5, ... for $n = 0, 1, 2, \ldots$. Similarly, the Fibonacci numbers (mod 7) repeat every 16 terms and F_{4n-1} repeats in the sequence 1, 2, 6, 5, ... for $n = 0, 1, 2, \ldots$. Thus, we have the following table:

$n \pmod 4$	0	1	2	3
$P_{3n+1} \pmod 7$	1	5	1	5
$(-1)^{\lfloor (n+1)/2 \rfloor} F_{4n-1} \pmod 7$	1	−2	−6	5

Since the given congruence holds in each case, it is true in general.

(a) A similar analysis proves part (a). Considering the sequences P_n and F_n modulo 13 suggests considering n modulo 28. A table of values for P_{3n-1} (mod 13) and F_{n+2} (mod 13) show that they repeat every 28 terms and the corresponding values are congruent.

Seiffert also found that $P_{6n-4} \equiv (-1)^{\lfloor (n-1)/2 \rfloor} F_{5n+2} \pmod{11}$. *He notes that many such congruences seem to exist.*

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, David C. Terr, David Zeitlin, and the proposer.

Another Pell Sum

B-758 ***Proposed by Russell Euler, Northwest Missouri State University, Maryville, MO (Vol. 32, no. 1, February 1994)***

Evaluate $\sum_{k=0}^{\infty} \frac{k2^k Q_k}{5^k}$.

Solution by Hans Kappus, Rodersdorf, Switzerland

Consider more generally

$$f(x) = \sum_{k=0}^{\infty} kQ_k x^k, \quad \text{for } |x| < \sqrt{2} - 1.$$

We will use the formula ([1], p. 21, formula 1.113)

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \quad \text{for } |x| < 1.$$

The Binet form for Q_k is $Q_k = (p^k + q^k)/2$, where $p = 1+\sqrt{2}$ and $q = 1-\sqrt{2}$. Substituting in the Binet form gives

$$f(x) = \frac{1}{2}\left\{\frac{px}{(1-px)^2} + \frac{qx}{(1-qx)^2}\right\} = \frac{x(1+2x-x^2)}{(1-2x-x^2)^2}.$$

In particular, since $2/5 < \sqrt{2} - 1$, the sum in question equals $f(2/5) = 410$.

Reference

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products.* San Diego, Calif.: Academic Press, 1980.

Also solved by Seung-Jin Bang, Glenn Bookhout, Paul S. Bruckman, Charles K. Cook, Bill Correll, Jr., Steve Edwards, Piero Filipponi, N. Gauthier, C. Georghiou, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Kostal, H. K. Krishnapriyan, Bob Prielipp, H.-J. Seiffert, Sahib Singh, David Zeitlin, and the proposer.

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