ON A PROBABILISTIC PROPERTY OF THE FIBONACCI SEQUENCE

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Let $\eta_1, \ldots, \eta_n, \ldots$ be a sequence of independent integer-valued random variables. Let $S_n = \eta_1 + \cdots + \eta_n$, $A_n = ES_n$, $B_n^2 = \operatorname{var} S_n$, $P_n(m) = P(S_n = m)$, and $f(t, \eta_j)$ denote the characteristic function of the random variable η_j.

The local limit theorem (LLT) is formulated as $P_n(m) = (2\pi B_n^2)^{-1/2} \cdot \exp\{-(m - A_n)^2 / 2B_n^2\} + o(B_n^{-1})$ when $n \to \infty$ uniformly for m.

The first results on the normal approximation of binomial distributions belong to de Moivre, Laplace, and Poisson. Very general theorems on the LLT were obtained by von Mises in [1]. Assuming additionally that the summands are i.i.d. and have a finite variance, B. Gnedenko [2] derived necessary and sufficient conditions for the LLT. The next step, for not i.i.d. but uniformly bounded variables, was made by Yu. V. Prohorov in [3]. Besides those mentioned above, the LLT problem was investigated by W. Feller [4] and C. Stone [5]. More complete bibliographical information can be found in [6].

It is well known that for uniformly distributed random variables the LLT is equivalent to the central limit theorem [9], [10]. Hence, it is reasonable to ask whether this holds in general. The answer is negative. Using the Fibonacci sequence, we will construct below another sequence of independent asymptotically uniformly distributed random variables which satisfies the central limit theorem, has the uniform asymptotic negligibility (UAN) property but for which the local limit theorem fails to be valid.

Let $[1; 1, \ldots, 1, \ldots]$ be a continued fraction representation of the number $\varphi = (1 + \sqrt{5})/2$. Denote by P_j / Q_j the convergents of the continued fraction of φ, which can be represented by the table below.

j					1	2	3	...
P_j	0	1	1	2	3	5	8	...
Q_j	1	0	1	1	2	3	5	...

It follows from the table that P_j $(j = 0, 1, 2, \ldots)$ is the Fibonacci sequence and $P_{j-1} = Q_j$ for $j \geq 1$.

Let us now consider a sequence of independent integer-valued random variables represented by

$$\begin{array}{ll} 1. & \xi_1, \ldots, \xi_{n_1}, \\ 2. & \xi_{n_1+1}, \ldots, \xi_{n_1+n_2}, \\ & \ldots \\ j. & \xi_{n_1+\cdots+n_{j-1}+1}, \ldots, \xi_{n_1+\cdots n_j}, \\ & \ldots \\ k. & \xi_{n_1+\cdots+n_{k-1}+1}, \ldots, \xi_{n_1+\cdots+n_k}. \end{array} \tag{1}$$

Each value of the line j is assumed to take the values 0, Q_j, P_j with respective probability values of $(P_j-2)/P_j, 1/P_j, 1/P_j$. Thus, if ξ_r is in row j, then

$$f(t,\xi_r)=\frac{P_j-2+e^{itQ_j}+e^{itP_j}}{P_j},$$

$$|f(t,\xi_r)|^2=\frac{(P_j-2)^2+2}{P_j^2}+\frac{2}{P_j^2}\cos t(P_j-Q_j)+\frac{2(P_j-2)}{P_j^2}(\cos tQ_j+\cos tP_j),$$

$$E\xi_r=\frac{(P_j+Q_j)}{P_j}=\frac{P_{j+1}}{P_j},$$

and

$$\operatorname{var}\xi_r=\frac{P_j^2+Q_j^2}{P_j}-\frac{(P_j+Q_j)^2}{P_j^2}.$$

Notice that

$$\operatorname{var}\xi_r>(1-1/P_j)\frac{P_j^2+Q_j^2}{P_j}>\frac{1}{3}\frac{P_j^2+Q_j^2}{P_j}.$$

We will take n_j as

$$n_j=\left[P_j^{3/2}\right]+1, \tag{2}$$

where $[a]$ represents the integer part of a.

Let $N_k=n_1+\cdots+n_k$ and

$$B_{N_k}^2=\operatorname{var}S_{N_k}=\sum_1^k\left(\left[P_j^{3/2}\right]+1\right)\operatorname{var}\xi_{N_k}=O(P_k^{5/2}).$$

First, we will verify that the sequence has the UAN property. For an arbitrary n, we can choose a number k such that $N_{k-1}<n\le N_k$. Hence,

$$\max_{1\le j\le n}\left|\xi_j-E\xi_j\right|\le P_k \text{ and } B_{N_k}^2\ge 3^{-1}\sum_1^k(P_j^2+Q_j^2)n_j/P_j.$$

Therefore,

$$\max_{1\le j\le n}\left|\xi_j-E\xi_j\right|\Big/B_n\le c/P_{k-1}^{1/4}\to 0 \text{ as } n\to\infty. \tag{3}$$

Here and in what follows c denotes a positive constant. However, the same symbol c may also stand for different constants. The preceding limit result is the UAN property. One may also check that Liapunov's condition,

$$\left(\sum_{j=1}^n E\left|\xi_j-E\xi_j\right|^{2+\delta}\right)^{1/(2+\delta)}\Big/B_n\to 0,$$

for some $\delta>0$, holds.

Next, we will investigate the property of the sequence being asymptotically uniformly distributed. We use the Dvoretzky-Wolfowitz test [8], which states that this is so if, for an arbitrary fixed $h>0$ and $z=1,2,\dots,h-1$, the characteristic function of the sums of the independent random variables tends to zero at the rational point $2\pi\alpha$, where $\alpha=z/h$.

It will be assumed, without loss of generality, that z and h are mutually prime numbers. Clearly,

$$\left|f(2\pi z/h,\xi_{N_j})\right|\le 1/P_j+\left|(P_j-2)/P_j+\exp(2\pi iQ_jz/h)/P_j\right|.$$

Assume Q_j is not a multiple of h. We can then write $zQ_j=mh+k$, $1\le k\le h-1$. Hence,

$$\begin{aligned}\left|(P_j-2)/P_j+\exp(2\pi izQ_j/h)/P_j\right| &\le \max_{1\le k\le h-1}\left|(P_j-2)/P_j+\exp(2\pi ik/h)/P_j\right|\\ &= \max_{1\le k\le h-1}\left|(P_j-3)/P_j+(1+\exp(2\pi ik/h))/P_j\right|\\ &=(P_j-3)/P_j+\max_{1\le k\le h-1}\left|1+\exp(2\pi ik/h)\right|/P_j\\ &\le (P_j-1-\rho)/P_j,\end{aligned}$$

where $\rho=\rho(h)=2(1-\cos(\pi/h))$. That is,

$$\left|f(2\pi z/h,\xi_{N_j})\right|\le 1-\rho/P_j.$$

Choosing n_j, by (2), we obtain

$$\left|f(2\pi z/h,\xi_{N_j})\right|^{2n_j}\le(1-\rho/P_j)^{2P_j}\le\exp(-2\rho).$$

The latter inequality holds only when Q_j is not a multiple of h. Let us count the number of such Q_j. Since $P_{j-1}Q_j-P_jQ_{j-1}=\pm1$, it follows that Q_{j-1} and Q_j are not simultaneously multiples of h. Therefore, there are at least $[k/2]$ members of the sequence $Q_1,\dots,Q_n$ that are not multiples of h. Thus,

$$\prod_{j=1}^{k}\left|f(2\pi z/h,\xi_{N_j})\right|^{2n_j}\le\exp\{-k\rho\}\to 0\text{ as }k\to\infty.$$

Therefore, the Dvoretsky-Wolfowitz test is satisfied.

It should be noted that in [7] we find the following necessary condition for the LLT:

$$I_n=B_n\int_{\varepsilon_n\le t\le 2\pi}\prod_{k=1}^{n}\left|f(t,\xi_j)\right|^2dt\to 0$$

for any positive ε_n which tends to zero as $n\to\infty$. We will show that this condition does not hold.

Using Taylor's expansion when $|t-2\pi/\varphi|\le 1/B_{N_k}$, we write

$$\begin{aligned}\left|f(t,\xi_{N_j})\right|^2 &= \left|f(2\pi/\varphi,\xi_{N_j})\right|^2+\left|t-2\pi/\varphi\right|\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=2\pi/\varphi}\\ &\quad+\left|t-2\pi/\varphi\right|^2\left[\left|f(t,\xi_{N_j})\right|^2\right]''_{t=\theta}\Big/2,\end{aligned}\tag{4}$$

where $t<\theta<2\pi/\varphi$.

Next, each term from (4) is estimated separately in the following manner:

$$\left|f(2\pi/\varphi,\xi_{N_j})\right|^2=\left|f(t_j,\xi_{N_j})\right|^2+\left|t_j-2\pi/\varphi\right|\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=t_j}$$

$$+\left|t_j-2\pi/\varphi\right|^2\left[\left|f(t,\xi_{N_j})\right|^2\right]''_{t=\theta}\Big/2,$$

where $t_j=2\pi Q_{j-1}/Q_j$.

Using $P_{j-1}Q_j-P_jQ_{j-1}=\pm1$ and the elementary inequality $\cos x\ge 1-x^2/2$, we may write

$$\begin{aligned}\left|f(t_j,\xi_{N_j})\right|^2&\ge\frac{(P_j-2)^2+2+2(P_j-2)}{P_j^2}+\frac{2+2(P_j-2)}{P_j^2}(1-1/2(2\pi/Q_j)^2)\\&=1-\frac{(P_j-1)}{P_j^2}(2\pi/Q_j)^2>1-(2\pi/Q_j)^2/P_j,\quad(j\ge2).\end{aligned}\tag{5}$$

We then have

$$\begin{aligned}\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=t_j}&=-\frac{2(P_j-Q_j)}{P_j^2}\sin 2\pi\frac{Q_{j-1}}{Q_j}(P_j-Q_j)-\frac{2Q_j(P_j-2)}{P_j^2}\\&\quad\times\sin 2\pi Q_{j-1}-\frac{2P_j(P_j-2)}{P_j}\sin 2\pi\frac{Q_{j-1}}{Q_j}P_j\\&=\left(\frac{2(P_j-Q_j)}{P_j^2}+\frac{2P_j(P_j-2)}{P_j^2}\right)\sin\left((-1)^{j-1}\frac{2\pi}{Q_j}\right)\le\left(\frac{2(P_j-Q_j)}{P_j^2}+\frac{2P_j(P_j-2)}{P_j^2}\right)\frac{2\pi}{Q_j}\\&=\frac{4\pi}{Q_j}(1-1/P_j-Q_j/P_j^2)\le 4\pi/Q_j\end{aligned}\tag{6}$$

and

$$\begin{aligned}\left[\left|f(t,\xi_{N_j})\right|^2\right]''&=2\left|(P_j-2)\cos P_jt+\frac{(P_j-2)Q_j^2}{P_j^2}\cos Q_jt+\frac{(P_j-Q_j)^2}{P_j^2}\cos(P_j-Q_j)t\right|<\left|f''(0,\xi_{N_j})\right|\\&=2\left((P_j-2)+\frac{(P_j-2)Q_j^2}{P_j^2}+\frac{(P_j-Q_j)^2}{P_j^2}\right)\\&=2\left(P_j-1+\frac{Q_j^2-2Q_j}{P_j}-\frac{Q_j^2}{P_j^2}\right)<2(P_j^2+Q_j^2)/P_j.\end{aligned}$$

Using $\left|t_j-2\pi/\varphi\right|\le 2\pi/Q_j^2$ and taking into consideration the estimations (5) and (6), we have

$$\left|f(2\pi/\varphi,\xi_{N_j})\right|^2\ge1-(2\pi/Q_j)^2/P_j-8\pi^2/Q_j^3-(2\pi/Q_j^2)^2(P_j^2+Q_j^2)/P_j.$$

Furthermore,

$$\left[\left|f(t,\xi_{N_j})\right|^2\right]'_{t=2\pi/\varphi} \le \left[\left|f(t_j,\xi_{N_j})\right|^2\right]' + \left|t_j - 2\pi/\varphi\right| \left[\left|f(t_j,\xi_{N_j})\right|^2\right]''$$
$$\le 4\pi/Q_j + 4\pi(P_j^2+Q_j^2)/Q_j^2 P_j$$

and

$$\left[\left|f(t,\xi_{N_j})\right|^2\right]''_{t=\theta} \le 2(P_j^2+Q_j^2)/P_j.$$

Taking the above estimations into account for expansion (4), we have

$$\left|f(t,\xi_{N_j})\right|^2 \ge 1-(2\pi/Q_j)^2 P_j - 8\pi^2/Q_j^3 - 8\pi^2(P_j^2+Q_j^2)/Q_j^4 P_j$$
$$-(4\pi/Q_j + 4\pi(P_j^2+Q_j^2)/Q_j^2 P_n)/B_{N_k} - 2(P_j^2+Q_j^2)/B_{N_k}^2 P_j.$$

By a simple transformation, we obtain

$$\left|f(t,\xi_{N_j})\right|^2 \ge 1 - c/P_j^3 - c/B_{N_k}P_j - cP_j/B_{N_k}^2.$$

Using the elementary inequality $\exp(-cx) < 1-x$ for $0 < x < 1/2$, and $c > \ln 4$, we have

$$\prod_{j=1}^{k}\left|f(t,\xi_{N_j})\right|^{2n_j} \ge \exp\left\{-\sum_{j=1}^{k} n_j(P_j^{-3} + (B_{N_k}P_j)^{-1} + P_j B_{N_k}^{-2})\right\}.$$

Hence, we conclude that, if k is sufficiently large, then

$$I_{N_k} > B_{N_k} \int\limits_{|t-2\pi/\varphi| \le B_{N_k}^{-1}} \prod_{j=1}^{n}\left|f(t,\xi_j)\right|^{2n_j} dt > B_{N_k} \int\limits_{|t-2\pi/\varphi|\le B_{N_k}^{-1}} \exp(-c)dt = 2e^{-c}.$$

So we have shown that the sequence (1) of independent integer valued random variables constructed by using the Fibonacci sequence is asymptotically uniformly distributed, satisfies the central limit theorem, and has the UAN property, but the local limit theorem fails to be valid for (1).

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