EXPONENTIAL GROWTH OF RANDOM FIBONACCI SEQUENCES

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1. INTRODUCTION

There are various ways in which the standard Fibonacci sequence can be generalized. Examples are:

1. Choose arbitrary starting values.
2. Introduce extra terms, for example, the "Tribonacci" sequence, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.
3. Introduce multipliers, for example, $x_n = ax_{n-1} + bx_{n-2}$, where a and b are positive integers or, more generally, positive (real) numbers.

A natural question to ask is: What is the rate of growth of the sequence? This could be tackled by investigating whether $x_n \sim K\phi^n$ for some constants K and ϕ, or the weaker condition, the convergence of $\frac{1}{n}\ln(x_n)$ as $n \to \infty$. If $\frac{1}{n}\ln(x_n)$ converges to ψ, then ψ is the rate of exponential growth in the sense that, for every $\delta > 0$,

$$\frac{x_n}{e^{(\psi+\delta)n}} \to 0 \quad \text{and} \quad \frac{x_n}{e^{(\psi-\delta)n}} \to \infty.$$

In this paper a further generalization of the Fibonacci sequence is considered. Instead of using fixed multipliers, choose pairs (a_n, b_n) at random, according to some specified probability distribution, and let

$$x_0 = 0,\ x_1 = 1,\ x_n = a_n x_{n-1} + b_n x_{n-2}, \quad n \geq 2.$$

$\{x_n\}$ is now a sequence of random variables.

A simple example is to choose a_n to be either 1 or 2 with probability $\frac{1}{2}$ (and independently of the previous a's) and to take all b_n's equal to 1.

We will show that, subject to certain conditions on the probability distribution of the multipliers, $\frac{1}{n}\ln(x_n)$ converges to a constant ψ for every sequence except for those in a set which together have zero probability of occurring.

2. MAIN RESULT

Let $\{a_n, b_n\}_{n\geq 1}$ be a sequence of pairs of random variables that satisfy the following conditions:

1. a_n and b_n are strictly positive.
2. (a_n, b_n) are independent pairs, that is, for every n and $k \geq 1$, and for all $0 \leq c_{n+j} < d_{n+j} < \infty$ and $0 \leq e_{n+j} < f_{n+j} < \infty$,

$$P(c_n < a_n \leq d_n,\ e_n < b_n \leq f_n, \ldots, c_{n+k} < a_{n+k} \leq d_{n+k},\ e_{n+k} < b_{n+k} \leq f_{n+k})$$
$$= P(c_n < a_n \leq d_n,\ e_n < b_n \leq f_n) \cdots P(c_{n+k} < a_{n+k} \leq d_{n+k},\ e_{n+k} < b_{n+k} \leq f_{n+k}).$$

This means that the probability distribution of (a_n, b_n) is not affected by knowing the values of the previous a's and b's.

3. $P(c < a_n \le d, e < b_n \le f) = P(c < a_1 \le d, e < b_1 \le f)$ for all n and for all $0 \le c < d < \infty$ and $0 \le e < f < \infty$.
4. $-\infty < E(\ln(a_1)) < \infty$ and $-\infty < E(\ln(b_1)) < \infty$.

$$\left[E(\ln(a_1)) = \int_0^\infty \ln(x)F(dx), \text{ where } F(x) = P(a_1 \le x). \text{ Similarly for } E(\ln(b_1)).\right]$$

Let

$$w_n = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{b_{n-1}}{a_n}}}}}.$$

w_n is a finite continued fraction (see Hardy and Wright [4] for basic properties).

Definition 1: To say that a condition holds on a sequence of random variables $\{z_n\}$ *almost surely* (a.s.) means that the sequences for which it does not hold form a set which has probability (measure) 0.

We will show that the sequence $\{w_n\}$ converges almost surely. Let w denote the limiting random variable.

Theorem 1: $\frac{1}{n}\ln(x_n) \xrightarrow{\text{a.s.}} \psi$, where $\psi = E(\ln(w))$.

Note: Since $a_1 < w < a_1 + \frac{b_1}{a_2}$, condition 4 implies that $E(\ln(w))$ is finite.

We note that the same method is used by Billingsley ([1], Ch. 1, §4) to prove a result of a similar nature involving the rate of growth of the "convergents" to a number by Diophantine approximation.

For $n \ge 2$,

$$x_n = a_n x_{n-1} + b_n x_{n-2} \quad \text{or} \quad \frac{x_n}{x_{n-1}} = a_n + b_n \frac{x_{n-2}}{x_{n-1}}.$$

Let

$$\begin{aligned} y_n &= \frac{x_n}{x_{n-1}}, \qquad n \ge 2, \\ &= a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{\ddots \, \cfrac{b_3}{a_2 + \cfrac{b_2}{a_1}}}} \end{aligned}$$

Let $y_1 = x_1$, then

$$\frac{1}{n}\ln(x_n) = \frac{1}{n}\sum_{k=1}^{n}\ln(y_i).$$

Proposition 1: $\{w_n\}$ converges almost surely.

Proof:

$$w_2 - w_1 = \frac{b_1}{a_2}$$

$$w_3 - w_2 = \frac{-b_1 b_2}{a_2(a_2 a_3 + b_2)}$$

$$w_4 - w_3 = \frac{b_1 b_2 b_3}{(a_2 a_3 + b_2)(a_2 a_3 a_4 + a_4 b_2 + a_2 b_3)}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Let

$$c_2 = 1,\ d_2 = a_2,$$
$$c_{n+1} = d_n,$$
$$d_{n+1} = a_{n+1} d_n + b_n c_n.$$

Then

$$w_n - w_{n-1} = \frac{(-1)^n b_1 b_2 \cdots b_{n-1}}{c_n d_n}.$$

A well-known property of continued fractions is that $\{w_{2n}\}$ is monotone decreasing and $\{w_{2n+1}\}$ is monotone increasing.

Ignoring all terms with two or more a's,

$$d_n \geq b_2 b_4 b_6 \cdots b_{n-1} \qquad \text{if } n \text{ is odd, } n \geq 3, \text{ while}$$

$$\begin{aligned} d_n \geq\ & a_n b_{n-2} b_{n-4} \cdots b_2 + a_{n-2} b_{n-1} b_{n-4} \cdots b_2 \\ & + a_{n-4} b_{n-1} b_{n-3} b_{n-6} \cdots b_2 + \cdots \\ & + a_2 b_{n-1} b_{n-3} \cdots b_3 \qquad \text{if } n \text{ is even, } n \geq 4. \end{aligned}$$

Hence, for n even, $|w_n - w_{n-1}|$ and $|w_{n+1} - w_n|$ are bounded by

$$\frac{1}{\frac{a_2}{b_1} + \frac{a_4}{b_3}\frac{b_2}{b_1} + \frac{a_6}{b_5}\frac{b_2 b_4}{b_1 b_3} + \cdots + \frac{a_n}{b_{n-1}}\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}}.$$

If every $b_n = 1$, this becomes $\frac{1}{a_2 + a_4 + a_6 + \cdots + a_n}$, which tends to 0 almost surely.

Otherwise, $\ln\left(\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}\right)$ is a symmetric random walk and, with probability one, will take values $\geq k$, for every k, for some value n. Thus, since the sequence $\left\{\frac{a_n}{b_{n-1}}\frac{b_2 b_4 \cdots b_{n-2}}{b_1 b_3 \cdots b_{n-3}}\right\}$ is unbounded almost surely, the denominator diverges almost surely

$$\text{or,} \quad |w_n - w_{n-1}| \xrightarrow{\text{a.s.}} 0.$$

Together with the fact that $\{w_{2n}\}$ and $\{w_{2n+1}\}$ are monotone, this implies that $\{w_n\}$ converges almost surely.

The ergodic theorem appears in many forms. In a probabilistic context it usually involves "stationary" sequences of random variables (see Billingsley [1], or Breiman [2], Ch. 6).

Definition 2: A sequence $\{z_n\}_{n\geq 1}$ of random variables is called *stationary* if $(z_1, z_2, \ldots, z_k)$ and $(z_{n+1}, z_{n+2}, \ldots, z_{n+k})$ have the same probability distribution for every $k \geq 1$ and $n \geq 1$.

The sequence $\{z_n\}$ determines a probability measure P on $(R^{\mathbb{Z}^+}, \mathscr{F})$, where $\mathscr{F}$ is the σ-field of events generated by $\{z_n\}$ (Breiman [2], Ch. 2).

Definition 3: A *tail event* A is one that does not depend on the values of $z_1, z_2, \ldots, z_n$ for any n. [For example, $A = (\{z_n\} : z_n \text{ converges})$ is a tail event.]

If every tail event has probability 0 or 1, then $\{z_n\}$ is *ergodic* (Breiman [2], Prop. 6.3.2 and Def. 6.30).

Consider now the doubly infinite sequence $\{a_n, b_n\}_{n\in\mathbb{Z}}$. For $n \geq 1$, let

$$z_n = \ln\left(a_n + \cfrac{b_n}{a_{n-1} + \cfrac{b_{n-1}}{a_{n-2} \cdots}}\right).$$

Proposition 2: $\{z_n\}$ is ergodic.

Proof: Stationarity is an immediate consequence of conditions 2 and 3.

A tail event for $\{z_n\}$ corresponds to an event that does not involve $\ldots(a_0, b_0), \ldots, (a_n, b_n)$ for every $n \geq 1$.

Since (a_n, b_n) are independent pairs, it can be deduced from Kolmogorov's 0–1 law (Breiman [2], Th. 3.12) that all tail events for $\{z_n\}$ have probability 0 or 1.

Theorem 2 (Ergodic Theorem):

$$\frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{\text{a.s.}} E(z_1).$$

Proof of Theorem 1:

$$\frac{1}{n}\ln(x_n) = \frac{1}{n}\sum_{i=1}^{n}\ln(y_i) \xrightarrow{\text{a.s.}} E(z_1)$$

$$\text{since } \frac{1}{n}\sum_{1}^{n} z_i \xrightarrow{\text{a.s.}} E(z_1)$$

$$\text{and } |\ln(y_n) - z_n| \xrightarrow{\text{a.s.}} 0.$$

3. EXAMPLE

Let $a_n = 1$, with probability 1/2 ; $a_n = 2$, with probability 1/2 (a_n's are chosen independently); $b_n \equiv 1$. Examples of possible sequences are:

(i) 0, 1, 1, 3, 4, 7, 18, 25, 68, ...,
(ii) 0, 1, 3, 7, 10, 17, 44, 105, 149, ...

w represents a "randomly" chosen number whose continued fraction expansion contains only 1's and 2's (every possible sequence in the first n places is equally likely, for every n).

$E(\ln(w))$ is easily approximated by

$$\frac{1}{2^n}\sum_{a=1\text{ or }2}\ln\left(a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots\,a_{n-1}+\cfrac{1}{a_n}}}}\right)$$

and is, to three decimal places, .673.

Hence, almost surely, such sequences grow at the rate $e^{.673}=1.960$.

This compares with a result of Davison [3] which was recently brought to the author's attention.

Let x be an irrational number in (0, 2).

Define $b_n=1+([nx]\bmod 2)$ ($[x]=$ integer part of x)

($\{b_n\}$ is a sequence of 1's and 2's).

Let $x_n=b_nx_{n-1}+x_{n-2}$.

Then $\lim_{n\to\infty}x_n^{1/n}$ always lies between 1.93 and 1.976.

4. CONCLUDING REMARKS

The conditions on (a_n, b_n) are not meant to be optimal. Any improvement, however, would result in greater complexity both of the results and proofs.

An interesting feature of the above results is that while individual sequences grow at a rate e^{ψ}, the average value of x_n. [$E(x_n)$, the expectation value], in general, grows at a different rate, since the sequence $\{E(x_n)\}$ satisfies $E(x_n)=E(a_n)E(x_{n-1})+E(b_n)E(x_{n-2})$; hence, $\frac{1}{n}\ln(E(x_n))$ $\to\ln\phi$, where ϕ is the positive root of $x^2-E(a_1)x-E(b_1)=0$ [assuming $E(a_1)$ and $E(b_1)$ are finite].

(For the example in Section 3, $\phi=2$.)

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