ON THE k^{th} DERIVATIVE SEQUENCES OF FIBONACCI AND LUCAS POLYNOMIALS*

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1. INTRODUCTION

As in [1] and [2], the Fibonacci polynomials $U_n(x)$ and the Lucas polynomials $V_n(x)$ (or simply U_n and V_n, when no misunderstanding can arise) are defined by the second-order linear recurrence relations

$$U_n = xU_{n-1} + U_{n-2} \quad (U_0 = 0,\ U_1 = 1), \tag{1}$$

and

$$V_n = xV_{n-1} + V_{n-2} \quad (V_0 = 2,\ V_1 = x), \tag{2}$$

where x is an indeterminate. Then the k^{th} derivatives of $U_n(x)$ and $V_n(x)$ are

$$U_n^{(k)} = \frac{d^k}{dx^k} U_n \quad \text{and} \quad V_n^{(k)} = \frac{d^k}{dx^k} V_n,$$

respectively. For convenience, we write $U_n^{(0)} = U_n$ and $V_n^{(0)} = V_n$.

Since $U_{-n} = (-1)^{n+1} U_n$ and $V_{-n} = (-1)^n V_n$, it can easily be deduced that the recurrence relations (1) and (2) hold for any integer n, and

$$U_{-n}^{(k)} = (-1)^{n+1} U_n^{(k)}, \tag{3}$$

$$V_{-n}^{(k)} = (-1)^n V_n^{(k)}. \tag{4}$$

The sequences $\{F_n^{(k)}\}$ and $\{L_n^{(k)}\}$ are defined as $F_n^{(k)} = [U_n^{(k)}(x)]_{x=1}$ and $L_n^{(k)} = [V_n^{(k)}(x)]_{x=1}$.

For $k = 1$ and 2, the sequences $\{U_n^{(k)}\}$, $\{V_n^{(k)}\}$, $\{F_n^{(k)}\}$, and $\{L_n^{(k)}\}$ were considered in [1] and [2], respectively. For any $k > 0$, the following conjectures were made in [2]:

Conjecture 1: $L_n^{(k)} = nF_n^{(k-1)}$.

Conjecture 2: $L_n^{(k)} = (n-k+1)L_n^{(k-1)} - 2(L_{n-1}^{(k)} + F_{n-1}^{(k-1)})$.

Conjecture 3: $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + kF_{n-1}^{(k-1)}$.

Conjecture 4: $L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + kL_{n-1}^{(k-1)}$.

Conjecture 5: $F_{n-1}^{(k)} + F_{n+1}^{(k)} = L_n^{(k)}$.

Conjecture 6: $F_n^{(k)} \equiv L_n^{(k)} \equiv 0 \pmod 2$ for $k \geq 2$.

Conjecture 7: $L_n^{(k)} \equiv 0 \pmod n$ for $k \geq 1$.

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The goal of this paper is to establish some identities and congruences involving the polynomials $U_n^{(k)}$ and $V_n^{(k)}$. For the sake of brevity, we shall not list the corresponding identities involving $F_n^{(k)}$ and $L_n^{(k)}$. The reader can easily obtain them by letting $x = 1$ in the general identities. The validity of the above conjectures emerges from the results established in Section 2. Observe that all results have been obtained by making no use of the explicit expressions for $U_n(x)$ and $V_n(x)$ which one can get by taking the k^{th} derivatives with respect to x of the sums (1.6) and (1.7) of [1], respectively.

2. SOME IDENTITIES AND CONGRUENCES INVOLVING $U_n(x)$ AND $V_n(x)$

The following four identities are most basic.

Identity 1: $U_{n-1}^{(k)} + U_{n+1}^{(k)} = V_n^{(k)}$ for $k \geq 0$.

Identity 2: $U_n^{(k)} = xU_{n-1}^{(k)} + U_{n-2}^{(k)} + kU_{n-1}^{(k-1)}$ for $k \geq 0$.

Identity 3: $V_n^{(k)} = xV_{n-1}^{(k)} + V_{n-2}^{(k)} + kV_{n-1}^{(k-1)}$ for $k \geq 0$.

Identity 4: $V_n^{(k)} = nU_n^{(k-1)}$ for $k \geq 1$.

Proof of Identity 1: That $U_{n-1} + U_{n+1} = V_n$ is a well-known fact. Take the k^{th} derivative (with respect to x) of both sides of this identity. □

Proof of Identity 2 (by induction on k***):*** The identity clearly holds for $k = 0$. Suppose it holds for a certain $k - 1 \geq 1$, that is, suppose that $U_n^{(k-1)} = xU_{n-1}^{(k-1)} + U_{n-2}^{(k-1)} + (k-1)U_{n-1}^{(k-2)}$. Take the first derivative of both sides of this identity. □

Identity 3 can be proved in a similar way.

Proof of Identity 4: Clearly, it suffices to prove that it holds for $k = 1$. This has been done in [1, formula (2.4)]. □

The following variety of identities can be regarded as generalizations of Identity 1.

Identity 5: $U_{n+m}^{(k)} + (-1)^m U_{n-m}^{(k)} = \dfrac{d^k}{dx^k}(U_n V_m) = \sum_{i=0}^{k} \binom{k}{i} U_n^{(i)} V_m^{(k-i)}$ for $k \geq 0$.

Identity 6: $U_{n+m}^{(k)} - (-1)^m U_{n-m}^{(k)} = \dfrac{d^k}{dx^k}(V_n U_m) = \sum_{i=0}^{k} \binom{k}{i} V_n^{(i)} U_m^{(k-i)}$ for $k \geq 0$.

Identity 7: $V_{n+m}^{(k)} + (-1)^m V_{n-m}^{(k)} = \dfrac{d^k}{dx^k}(V_n V_m) = \sum_{i=0}^{k} \binom{k}{i} V_n^{(i)} V_m^{(k-i)}$ for $k \geq 0$.

Identity 8: $V_{n+m}^{(k)} - (-1)^m V_{n-m}^{(k)} = \dfrac{d^k}{dx^k}(U_n W_m) = \dfrac{d^k}{dx^k}(W_n U_m)$ for $k \geq 0$.

Here and in the sequel to this paper, we let $W_n = V_{n-1} + V_{n+1} = (x^2 + 4)U_n$.

Evidently, the above four identities follow immediately from the case $k = 0$ for which we have the following well-known results.

Identity 5': $U_{n+m}+(-1)^m U_{n-m}=U_n V_m$.

Identity 6': $U_{n+m}-(-1)^m U_{n-m}=V_n U_m$.

Identity 7': $V_{n+m}+(-1)^m V_{n-m}=V_n V_m$.

Identity 8': $V_{n+m}-(-1)^m V_{n-m}=U_n W_m=W_n U_m$.

To prove Conjectures 1-7, we shall establish another identity and two congruences.

Identity 9: $xV_n^{(k)}=(n-k+1)V_n^{(k-1)}-2(V_{n-1}^{(k)}+U_{n-1}^{(k-1)})$.

Proof: Using Identities 4, 1, and 3, we have

$$\begin{aligned}&(n-k+1)V_n^{(k-1)}-2(V_{n-1}^{(k)}+U_{n-1}^{(k-1)})\\&=(n+1)V_n^{(k-1)}-kV_n^{(k-1)}-2nU_{n-1}^{(k-1)}\\&=(n+1)V_n^{(k-1)}+xV_n^{(k)}+V_{n-1}^{(k)}-V_{n+1}^{(k)}-2nU_{n-1}^{(k-1)}\\&=xV_n^{(k)}+(n+1)V_n^{(k-1)}+(n-1)U_{n-1}^{(k-1)}+(n+1)U_{n+1}^{(k-1)}-2nU_{n-1}^{(k-1)}\\&=xV_n^{(k)}+(n+1)(V_n^{(k-1)}-U_{n-1}^{(k-1)}-U_{n+1}^{(k-1)})\\&=xV_n^{(k)}.\end{aligned}$$

Congruence 1: $U_n^{(k)}\equiv V_n^{(k)}\equiv 0 \pmod{k!}$.

Proof: If we take the k^{th} derivative with respect to x of the combinatorial sums which give U_n and V_n (e.g., see [1, (1.6) and (1.7)]), we see that each of their summands contains the product of k consecutive integers. It follows that all of them are divisible by $k!$. □

Congruence 2: $V_n^{(k)}\equiv 0 \pmod{n}$ for $k\ge 1$.

Proof: It is an immediate consequence of Identity 4.

Letting $x=1$ in the above stated identities and congruences yields the following corollary.

Corollary: Conjectures 1-7 are all true.

3. SOME CONVOLUTION IDENTITIES INVOLVING $U_n(x)$ AND $V_n(x)$

In this section we discuss some finite series involving $U_n^{(k)}$ and $V_n^{(k)}$ that have simple closed-form expressions for their sums.

Proposition 1: $\displaystyle\sum_{i=0}^{n}U_i^{(k)}U_{n-i}=\frac{1}{k+1}U_n^{(k+1)}$.

Proposition 2: $\displaystyle\sum_{i=0}^{n}U_i^{(k)}V_{n-i}=\frac{1}{k+1}V_n^{(k+1)}+U_n^{(k)}$.

Proposition 3: $\displaystyle\sum_{i=0}^{n}V_i^{(k)}U_{n-i}=\frac{1}{k+1}V_n^{(k+1)}+\delta(0,k)U_n$.

Proposition 4: $\sum_{i=0}^{n} V_i^{(k)} V_{n-i} = \frac{1}{k+1} W_n^{(k+1)} + (1+\delta(0,k)) V_n^{(k)}$.

Here, $\delta(0, k)$ is Kronecker's symbol which equals 1 if $k = 0$, and equals 0 otherwise.

Proofs of Propositions 1-4: Let $A_n^{(k)} = \sum_{i=0}^{n} U_i^{(k)} U_{n-i}$. Since U_j $(j \geq 1)$ is a monic polynomial of degree $j-1$ (cf. [2, (1.6)]), we have that $U_0^{(k)} = U_1^{(k)} = \cdots = U_k^{(k)} = 0$ and $U_{k+1}^{(k)} = k!$, so that $A_k^{(k)} = A_{k+1}^{(k)} = 0$ and $A_{k+2}^{(k)} = U_{k+1}^{(k)} U_1 = k! = \frac{1}{k+1} U_{k+2}^{(k+1)}$. Suppose that $A_{n-1}^{(k)} = \frac{1}{k+1} U_{n-1}^{(k+1)}$ and $A_{n-2}^{(k)} = \frac{1}{k+1} U_{n-2}^{(k+1)}$ for $n \geq 2$. Then

$$\begin{aligned} A_n^{(k)} &= \sum_{i=0}^{n} U_i^{(k)} U_{n-i} = \sum_{i=0}^{n-1} U_i^{(k)} (x U_{n-1-i} + U_{n-2-i}) = x A_{n-1}^{(k)} + A_{n-2}^{(k)} + U_{n-1}^{(k)} U_{-1} \\ &= \frac{1}{k+1}\left(x U_{n-1}^{(k+1)} + U_{n-2}^{(k+1)} + (k+1) U_{n-1}^{(k)}\right) = \frac{1}{k+1} U_n^{(k+1)}. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k)} V_{n-i} &= \sum_{n=0}^{n} U_i^{(k)} (U_{n-1-i} + U_{n+1-i}) = A_{n-1}^{(k)} + A_{n+1}^{(k)} + U_n^{(k)} U_{-1} \\ &= \frac{1}{k+1}\left(U_{n-1}^{(k+1)} + U_{n+1}^{(k+1)}\right) + U_n^{(k)} = \frac{1}{k+1} V_n^{(k+1)} + U_n^{(k)}. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} V_i^{(k)} U_{n-i} &= \sum_{i=0}^{n} (U_{i-1}^{(k)} + U_{i+1}^{(k)}) U_{n-i} = \sum_{i=1}^{n} U_{i-1}^{(k)} U_{n-i} + U_{-1}^{(k)} U_n + \sum_{i=0}^{n} U_{i+1}^{(k)} U_{n-i} \\ &= A_{n-1}^{(k)} + A_{n+1}^{(k)} + U_1^{(k)} U_n = \frac{1}{k+1} (U_{n-1}^{(k+1)} + U_{n+1}^{(k+1)}) + \delta(0,k) U_n \\ &= \frac{1}{k+1} V_n^{(k+1)} + \delta(0,k) U_n. \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} V_i^{(k)} V_{n-i} &= \sum_{n=0}^{n} V_i^{(k)} (U_{n-1-i} + U_{n+1-i}) = \sum_{i=0}^{n-1} V_i^{(k)} U_{n-1-i} + V_n^{(k)} U_{-1} + \sum_{i=0}^{n+1} V_i^{(k)} U_{n+1-i} \\ &= \frac{1}{k+1} (V_{n-1}^{(k+1)} + V_{n+1}^{(k+1)}) + \delta(0,k)(U_{n-1} + U_{n+1}) + V_n^{(k)} \\ &= \frac{1}{k+1} W_n^{(k+1)} + (1+\delta(0,k)) V_n^{(k)}. \end{aligned}$$

Furthermore, for any $k, j \geq 0$, we have

Proposition 5: $\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+1)}$.

Proposition 6: $\sum_{i=0}^{n} V_i^{(k)} U_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} V_n^{(k+j+1)} + \delta(0,k) U_n^{(j)}$.

Proposition 7: $\sum_{i=0}^{n} V_i^{(k)} V_{n-i}^{(j)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} W_n^{(k+j+1)} + (\delta(0,k) + \delta(0,j)) V_n^{(k+j)}$.

For the sake of brevity, we shall prove only Proposition 5.

Proof of Proposition 5 (by induction on j): By virtue of Proposition 1, the statement holds for $j = 0$. Suppose it holds for some $j \geq 1$. Since

$$\frac{d}{dx}\left(\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j)}\right) = \sum_{i=0}^{n} U_i^{(k+1)} U_{n-i}^{(j)} + \sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j+1)} = \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+2)},$$

we can write

$$\begin{aligned}\sum_{i=0}^{n} U_i^{(k)} U_{n-i}^{(j+1)} &= \left[(k+j+1)\binom{k+j}{j}\right]^{-1} U_n^{(k+j+2)} - \left[(k+1+j+1)\binom{k+1+j}{j}\right]^{-1} U_n^{(k+1+j+1)} \\ &= \left[(k+j+2)\binom{k+j+1}{j+1}\right]^{-1} \left[\frac{k+j+2}{j+1} - \frac{k+1}{j+1}\right] U_n^{(k+j+2)} \\ &= \left[(k+j+2)\binom{k+j+1}{j+1}\right]^{-1} U_n^{(k+j+2)}.\end{aligned}$$

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