ON A SYSTEM OF SEQUENCES DEFINED BY A RECURRENCE RELATION

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1. INTRODUCTION

Sequences defined by recurrence relations have been studied in many papers. Some of these studies treated the system of sequences defined by a recurrence relation. For instance, Lucas [6] studied the second-order case; Shannon and Horadam [8] dealt with the third-order recurrence relations.

The purpose of this note is to summarize some properties of the system of m sequences $\{u_{k,n}\}$ (where $k = 1, 2, \ldots, m$) defined by the recurrence relation

$$u_{k,n} = P_1 u_{k,n-1} + P_2 u_{k,n-2} + \cdots + P_m u_{k,n-m} \tag{1}$$

with initial conditions

$$u_{k,n} = \delta_{k,n+1} \text{ (where } k = 1, 2, \ldots, m;\ n = 0, 1, \ldots, m-1), \tag{2}$$

where the right-hand side stands for Kronecker's delta.

We will first write down the fundamental relations, and then consider the calculation of $u_{k,n}$. Finally, we will deal with some applications.

2. FUNDAMENTAL RELATIONS

A few leading terms for each of these sequences can be found in the following table:

$k \backslash n$	0	1	2	$\cdots$	$\cdots$	$\cdots$	$m-1$	m	$m+1$	$m+2$
1	1	0	0	$\cdots$	$\cdots$	$\cdots$	0	P_m	P_1P_m	$P_m(P_1^2+P_2)$
2	0	1	0	$\cdots$	$\cdots$	$\cdots$	0	P_{m-1}	$P_m + P_1P_{m-1}$	$P_{m-1}(P_1^2+P_2)+P_1P_m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	0	$\cdots$	$\cdots$	$\cdots$	0	1	0	P_2	$P_3 + P_1P_2$	$P_2(P_1^2+P_2)+P_1P_3+P_4$
m	0	$\cdots$	$\cdots$	$\cdots$	0	0	1	P_1	$P_2 + P_1^2$	$P_1(P_1^2+P_2)+P_1P_2+P_3$

Now, the fundamental relations

$$u_{1,n+1} = P_m u_{m,n},\ u_{k,n+1} = u_{k-1,n} + P_{m-k-1} u_{m,n} \text{ (for } k = 2, \ldots, m) \tag{3}$$

can be established easily by induction.

Using the matrices

$$U_n = \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ \vdots \\ u_{m-1,n} \\ u_{m,n} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & \cdots & 0 & P_m \\ 1 & 0 & \cdots & 0 & P_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & P_2 \\ 0 & 0 & \cdots & 1 & P_1 \end{pmatrix},$$

these relations can be written as

$$U_{n+1} = TU_n \tag{4}$$

which gives us

$$U_n = T^n U_0, \ T^n = (U_n, U_{n+1}, U_{n+2}, \ldots, U_{n+m-1}). \tag{5}$$

The generating functions

$$G_k(x) = u_{k,0} + u_{k,1}x + u_{k,2}x^2 + \cdots + u_{k,n}x^n + \cdots \ (\text{where } k = 1, 2, \ldots, m)$$

for these sequences are given by

$$G_k(x) = x^{k-1}H_k(x) / H_0(x), \tag{6}$$

where $H_k(x) = 1 - P_1x - P_2x^2 - \cdots - P_{m-k}x^{m-k}$ for $k = 0, 1, \ldots, m-1$ and $H_m(x) = 1$.

3. CALCULATION OF TERMS

From the generating function (6), we can easily determine the formula for $u_{m,n}$, which is

$$u_{m,m+n-1} = \sum_{s=1}^{n} \sum_{\substack{r_1+r_2+\cdots+r_s=n \\ 1\le r_i \le m \ (i=1,2,\ldots,s)}} P_{r_1}P_{r_2}\cdots P_{r_s},$$

where the summation runs over all the decompositions of n into the integers not exceeding m.

Following the method of Shannon and Horadam [7], it is easy to see that

$$u_{m,m+n-1} = \sum_{i_m=0}^{[t_m/m]} \cdots \sum_{i_3=0}^{[t_3/3]} \sum_{i_2=0}^{[t_2/2]} \frac{(t_1 + i_2 + \cdots + i_m)!}{t_1! i_2! \cdots i_m!} P_1^{t_1} P_2^{i_2} \cdots P_m^{i_m},$$

where $t_m = n$, and $t_k = t_{k+1} - (k+1)i_{k+1}$ for $k = m-1, \ldots, 2, 1$.

If the coefficients $P_1, P_2, \ldots, P_m$ are given numerically, we have an $O(m^2 \log n)$ algorithm for computing U_n by using (4) and (5). To see this, examine Er [2] or Gries and Levin [4].

In the case of $P_m \neq 0$, we can also define $u_{k,n}$ and U_n for negative values of n by using the recurrence relation (1) in the opposite direction. Formulas (4) and (5) are also valid for negative n and, in fact,

$$T^{-1} = \begin{pmatrix} Q_m & 1 & 0 & \cdots & 0 \\ Q_{m-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_2 & 0 & 0 & \cdots & 1 \\ Q_1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $Q_1 = P_m^{-1}$ and $Q_k = -P_{k-1}P_m^{-1}$ for $k = 2, 3, \ldots, m$. Thus, we have a similar algorithm for computing U_n when n is negative.

Following the ideas of Barakat [1] who used 2×2 matrices, we obtain similar formulas for the $m \times m$ matrices.

Theorem 1: Let X be an $m \times m$ matrix that has the characteristic equation

$$\lambda^m - P_1\lambda^{m-1} - \cdots - P_{m-1}\lambda - P_m = 0. \tag{7}$$

Then we have

$$X^n = u_{m,n}X^{m-1} + u_{m-1,n}X^{m-2} + \cdots + u_{2,n}X + u_{1,n}E, \tag{8}$$

where the coefficients are defined by (1) and (2).If X is regular, then $P_m \neq 0$, so that (8) is valid even for negative values of n.

Proof: For $n = 0, 1, \ldots, m-1$, (8) is valid by (2). On the other hand, we have

$$X^m = P_1X^{m-1} + P_2X^{m-2} + \cdots + P_{m-1}X + P_mE \tag{9}$$

by (7). Using this equality, we complete the proof for positive n by induction. To prove (8) for negative n, we use (1) and (9) in the opposite direction.

Next, we consider the evaluation of some series related to the sequences $\{u_{k,n}\}$.

Theorem 2: Let

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots \tag{10}$$

be a function defined by the power series in x that has the radius of convergence R with

$$R > \max_{1 \le i \le m} |\lambda_i|, \tag{11}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the characteristic equation

$$\varphi(\lambda) = \lambda^m - P_1\lambda^{m-1} - \cdots - P_{m-1}\lambda - P_m = 0$$

of the difference equations given in (1). For the sequences $\{u_{k,n}\}$ defined by (1) and (2), we consider the series

$$f_k = \sum_{n=0}^{\infty} c_n u_{k,n} \quad (k = 1, \ldots, m). \tag{12}$$

If all the λ_i's are distinct, f_k has the representation

$$f_k = (-1)^{m-k} \sum_{i=1}^{m} f(\lambda_i) \frac{p_{m-k,i}}{\varphi'(\lambda_i)} \quad (k = 1, 2, \ldots, m), \tag{13}$$

where $p_{0,i} = 1$, and $p_{h,i}$ stands for the elementary symmetric polynomial of degree h in $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m$, for $1 \le h \le m-1$.

Proof: Using (8) for $X = T$ as defined in (4), we have

$$f(T) = \sum_{n=0}^{\infty} c_n \sum_{k=1}^{m} u_{k,n}T^{k-1} = \sum_{k=1}^{m} f_k T^{k-1}.$$

If we compare this formula with Sylvester's formula (see Frazer et al. [3])

$$f(T)=\sum_{i=1}^{m} f(\lambda_i)\frac{(T-\lambda_1E)\cdots(T-\lambda_{i-1}E)(T-\lambda_{i+1}E)\cdots(T-\lambda_mE)}{(\lambda_i-\lambda_1)\cdots(\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1})\cdots(\lambda_i-\lambda_m)},$$

we have (13), since the minimal polynomial of T is of degree m in this case.

Applying this formula for $m=2$ and $m=3$ to the functions e^x, $\sin x$, $\cos x$, $\sinh x$, and so on, we obtain many of the formulas shown in [1], [7], [8], and [9].

If $\lambda_j=\lambda_i$, the formula for f_k corresponding to (13) will be given by taking the limit as λ_j tends to λ_i in (13).

For the sequence $\{u_n\}$ satisfying the same recurrence as (1) with initial conditions $u_n=a_n$ for $k=0,1,\ldots,m-1$, we have

$$\sum_{n=0}^{\infty} c_n u_n = \sum_{k=1}^{m} a_{k-1} f_k,$$

which can also be calculated directly from the general solution.

J. Z. and J. S. Lee [5] applied this latter method to a function $f(x)$ that had a geometric progression for its coefficients in order to characterize some B-power fractions.

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