ON CERTAIN ARITHMETIC PROPERTIES OF FIBONACCI AND LUCAS NUMBERS

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0. INTRODUCTION

In [1] the authors discussed, inter alia, certain striking resemblances between the arithmetic behavior of Fibonacci and Lucas numbers on the one hand, and certain numbers arising from (generalized) paper-folding on the other. To describe the latter, let t, u be mutually prime positive integers (we may assume $t > u$) and set

$$M_n = \frac{t^n - u^n}{t - u}, \quad P_n = t^n + u^n, \quad n \geq 1. \tag{0.1}$$

Then the sequence $\{M_n\}$ shares arithmetical properties with the Fibonacci sequence $\{F_n\}$, while the sequence $\{P_n\}$ is similarly related to the Lucas sequence $\{L_n\}$. In particular, we have

$$\gcd(M_a, M_b) = M_d, \text{ where } d = \gcd(a, b), \tag{0.2}$$

mirroring a well-known feature of Fibonacci numbers (see Theorem 2.5).

It was pointed out in [1] that (0.2) could itself be used to *disprove* the corresponding assertion for the lcm; precisely, if $\operatorname{lcm}(a, b) = \ell$, then $\operatorname{lcm}(M_a, M_b) = M_\ell$ only in the trivial cases $a|b$ or $b|a$. The argument rested on a uniqueness theorem for the expression of rational numbers as a ratio of two members of the $\{M_n\}$ sequence. However, the authors did not establish the corresponding negative results for $\operatorname{lcm}(F_a, F_b)$, $\operatorname{lcm}(L_a, L_b)$.

In this paper the gap is filled, precisely by establishing the relevant uniqueness statements for ratios of Fibonacci numbers and Lucas numbers. It turns out that much of the work can be done for arbitrary sequences $\{u_n\}$ of positive integers satisfying the recurrence relation $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$.

Such sequences are, in a sense, *classified* by their initial values u_1, u_2. However, to discuss the classification, it is better to take the sequences backward with respect to n, that is, to allow n to take *any* integer value, although the principal results are all to be concerned with positive values of n. Then the Fibonacci sequence $\{F_n\}$ belongs to the special class given by $u_0 = 0$. Another interesting class, from our point of view, is given by $0 < u_0 \leq u_1$. The Lucas sequence $\{L_n\}$ seems, to us, to belong to a singleton class.

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It is interesting to note that the Fibonacci sequence plays a special role in the study of the whole class of sequences $\{u_n\}$ (see Theorem 1.1).

This note closes with a related result concerning the lcm in the mixed case, i.e., lcm(F_a, L_b). The corresponding result for gcd(F_a, L_b) (see Theorem 2.9) is to be found in [1] and [2]. It would be interesting to seek the analogous result concerning gcd(M_a, P_b).

We use (r, s) for the open interval $r < x < s$ and $(r, s]$ for the half-open interval $r < x \le s$.

We do not claim originality for the statements of Corollary 2.3 and Theorem 2.4—though we have not ourselves succeeded in finding them in the literature. However, we do believe that our results on the lcm are entirely new.

1. GENERAL PROPERTIES

We assume here that we have a sequence $\{u_n, n \ge 0\}$ of integers such that $u_0 \ge 0$, $u_1 > 0$, and

$$u_{n+2} = u_{n+1} + u_n, \quad n \ge 0. \tag{1.1}$$

Of course, both the Fibonacci numbers $\{F_n\}$ and the Lucas numbers $\{L_n\}$ meet these conditions. We prove the following:

Theorem 1.1: Let $\ell \ge 1$. Then

$$\frac{u_{k+\ell}}{u_k} \in (F_{\ell+1}, F_{\ell+2}), \; k \ge 3; \; \frac{u_{2+\ell}}{u_2} \in (F_{\ell+1}, F_{\ell+2}].$$

Proof: We argue by induction on ℓ. If $\ell = 1$, then

$$\frac{u_{k+1}}{u_k} > 1 \text{ if } k \ge 2; \; \frac{u_2}{u_1} \ge 1.$$

On the other hand, $u_{k+1}/u_k = 1 + u_{k-1}/u_k < 2$ if $k \ge 3; u_3/u_2 = 1 + u_1/u_2 \le 2$. Hence

$$\frac{u_{k+1}}{u_k} \in (1, 2) = (F_2, F_3), \; k \ge 3; \; \frac{u_3}{u_2} \in (1, 2] = (F_2, F_3].$$

Now let $\ell = 2$. Then

$$\frac{u_{k+2}}{u_k} = \frac{u_{k+1}}{u_k} + 1 \in (2, 3) = (F_3, F_4), \; k \ge 3; \text{ and } \frac{u_4}{u_2} = \frac{u_3}{u_2} + 1 \in (2, 3] = (F_3, F_4].$$

We now carry out the inductive step. We assume the theorem is true for $\ell - 1, \ell - 2, \ell \ge 3$. Then

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{k+\ell-1}}{u_k} + \frac{u_{k+\ell-2}}{u_k} \in (F_\ell, F_{\ell+1}) + (F_{\ell-1}, F_\ell)$$
$$= (F_\ell + F_{\ell-1}, F_{\ell+1} + F_\ell) = (F_{\ell+1}, F_{\ell+2}), \text{ if } k \ge 3;$$

and

$$\frac{u_{2+\ell}}{u_2} = \frac{u_{1+\ell}}{u_2} + \frac{u_\ell}{u_2} \in (F_\ell, F_{\ell+1}] + (F_{\ell-1}, F_\ell]$$
$$= (F_\ell + F_{\ell-1}, F_{\ell+1} + F_\ell] = (F_{\ell+1}, F_{\ell+2}],$$

proving the theorem.

Remarks:

- ***(a)*** Notice that, for the Fibonacci numbers, $F_3 / F_2 = 2 = F_3$. Indeed $u_3 / u_2 = F_3$ precisely when $u_0 = 0$.
- ***(b)*** There is no need in this theorem for u_n to be an integer.

Theorem 1.2: $u_{k+\ell} = F_\ell u_{k+1} + F_{\ell-1} u_k$, for all k, ℓ.

Proof: We hold k fixed and prove this for two successive values of ℓ; plainly, this suffices. Now $F_{-1} = 1, F_0 = 0, F_1 = 1$, so it is plain that the formula holds* for $\ell = 0, 1$.

Theorem 1.3: Let $\frac{u_{k+1}}{u_k} = \frac{u_{m+1}}{u_m}$ for some positive k, m. Then $k = m$.

Proof: Plainly,

$$\frac{u_{k+1}}{u_k} = \frac{u_{m+1}}{u_m} \Leftrightarrow \frac{u_{k-1}}{u_k} = \frac{u_{m-1}}{u_m} \Leftrightarrow \frac{u_k}{u_{k-1}} = \frac{u_m}{u_{m-1}}.$$

Thus, if $k \neq m$, we have $h \geq 3$ such that

$$\frac{u_h}{u_{h-1}} = \frac{u_2}{u_1}. \tag{1.2}$$

We now proceed backwards, that is, we allow negative values of n in u_n. We look at the sequence $\{u_n, 2 \geq n > -\infty\}$. There are three possibilities:

- ***(i)*** As n decreases, this sequence remains positive. This, however, is impossible, since, if all terms are positive, it follows from (1.1) that the sequence is decreasing; but there is no strictly decreasing infinite sequence of positive integers.
- ***(ii)*** As n decreases, this sequence remains positive until it takes the value 0.
- ***(iii)*** As n decreases, this sequence remains positive until it takes a negative value.

Thus there must be a first value n for which u_n is nonpositive (as *n decreases*). It follows from (1.2) that

$$\frac{u_{r+1}}{u_r} = \frac{u_{n+1}}{u_n} \tag{1.3}$$

with $r > n$. Thus $u_{r+1} u_n = u_r u_{n+1}$, with u_{r+1}, u_r, u_{n+1} positive and u_n nonpositive. This contradiction implies $k = m$.

From Theorems 1.2 and 1.3, we infer

Theorem 1.4: Let $\frac{u_{k+\ell}}{u_k} = \frac{u_{m+\ell}}{u_m}$ for some positive k, m, and $\ell \geq 1$. Then $k = m$.

Proof: By Theorem 1.2, we infer that

$$\frac{F_\ell u_{k+1}}{u_k} + F_{\ell-1} = \frac{F_\ell u_{m+1}}{u_m} + F_{\ell-1}.$$

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* We may, of course, continue the sequence $\{u_n\}$ backwards, using (1.1). In particular, we may define F_n, L_n for n negative.

Since $\ell \geq 1, F_\ell \neq 0$, so $u_{k+1} / u_k = u_{m+1} / u_m$. We apply Theorem 1.3.

We now put together Theorems 1.4 and 1.1 to infer

Theorem 1.5: If $\frac{u_{k+\ell}}{u_k} = \frac{u_{m+n}}{u_m}$ with k, $m \geq 2$, and $\ell, n \geq 1$, then $k = m, \ell = n$.

Proof: By Theorem 1.1,

$$\frac{u_{k+\ell}}{u_k} \in (F_{\ell+1}, F_{\ell+2}], \quad \frac{u_{m+n}}{u_m} \in (F_{n+1}, F_{n+2}].$$

Now if $\ell, n \geq 1$, then $(F_{\ell+1}, F_{\ell+2}]$ and $(F_{n+1}, F_{n+2}]$ are disjoint unless $\ell = n$. Thus, $\ell = n$ and

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+\ell}}{u_m}, \ \ell \geq 1, \text{ so that, by Theorem 1.4, } k = m.$$

Remark: In fact, it is only in the proof of Theorem 1.3 that we use the fact that $\{u_n\}$ is a sequence of integers. However, this, of course, implies that Theorems 1.4 and 1.5 are only proved under this assumption. Note that the conclusion of Theorem 1.3 is false if $u_n = \phi^n$, where ϕ is the golden section (so that $\phi^2 = \phi + 1$).

2. SPECIAL PROPERTIES OF FIBONACCI AND LUCAS NUMBERS

It is noteworthy that, in Theorem 1.5, we must exclude the possibility that $k = 1$ or $m = 1$. Of course, if we require $\ell, n \geq 1$, then the conclusion of Theorem 1.5 trivially follows if $k = m = 1$, for then $u_{\ell+1} = u_{n+1}$ with $\ell + 1, n + 1 \geq 2$, and the sequence $\{u_n, n \geq 2\}$ is strictly increasing. However, we also have to consider the possibilities $k = 1, m \geq 2$ and $m = 1, k \geq 2$.

In considering these possibilities, we are content largely to confine our attention to the sequences $\{F_n\}$ and $\{L_n\}$ of Fibonacci and Lucas numbers, since it is to arithmetic properties of these sequences that we will be applying our enhanced form of Theorem 1.5. However, notice that we get the enhanced form for certain sequences $\{u_n\}$ immediately, by the following observation.

Suppose that, in fact, $0 < u_0 \leq u_1$. Set $v_n = u_{n-1}, n \geq 0$, noticing that $u_{-1} = u_1 - u_0 \geq 0$. Thus we may apply Theorem 1.5 to the sequence $\{v_n\}$, obtaining

$$\frac{v_{k+\ell}}{v_k} = \frac{v_{m+n}}{v_m}, \text{ with } k, m \geq 2, \ \ell, n \geq 1 \Rightarrow k = m, \ \ell = n. \tag{2.1}$$

Rewrite (2.1) in terms of the original sequence $\{u_n\}$, writing $k+1$ for k, $m+1$ for m; we obtain

Theorem 2.1: If, in addition, $0 < u_0 \leq u_1$, then

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+n}}{u_m}, \text{ with } k, m \geq 1, \ \ell, n \geq 1 \Rightarrow k = m, \ \ell = n. \tag{2.2}$$

We must proceed differently in seeking the enhanced form of Theorem 1.5 in the case of the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. For, of course, neither sequence satisfies

$0 < u_0 \le u_1$. Indeed, $F_0 = 0$ and $L_0 > L_1$. Thus we first attach the supplementary condition $u_0 = 0$; this means that $u_n = u_1 F_n$. We prove

Theorem 2.2: If, in addition, $u_0 = 0$, then

$$\frac{u_{k+\ell}}{u_k} = \frac{u_{m+n}}{u_m}, \text{ with } k, m \ge 1,\ \ell, n \ge 1 \Rightarrow k = m,\ \ell = n;$$

$$\text{or } \quad k = 1,\ m = 2,\ \ell = n+1;$$

$$\text{or } \quad k = 2,\ m = 1,\ n = \ell + 1.$$

Proof: We may assume $k = 1, m \ge 2$ in light of Theorem 1.5 and the opening remark of this section. Then

$$\frac{u_{\ell+1}}{u_1} = \frac{u_{m+n}}{u_m} \text{ or } F_{\ell+1} = \frac{F_{m+n}}{F_m}.$$

But $m \ge 2$ so that, by Theorem 1.1,

$$F_{\ell+1} = \frac{F_{m+n}}{F_m} \in (F_{n+1}, F_{n+2}],\ \ell, n \ge 1.$$

This forces $\ell = n+1$, so that

$$\frac{F_{m+n}}{F_m} \notin (F_{n+1}, F_{n+2}),$$

forcing $m = 2$. Of course, the case $m = 1, k \ge 2$ is treated similarly.

Applying Theorem 2.2 to the sequence $\{F_n\}$, we have

Corollary 2.3: If

$$\frac{F_{k+\ell}}{F_k} = \frac{F_{m+n}}{F_m}, \text{ with } k, m \ge 1,\ \ell, n \ge 1 \text{ then } k = m,\ \ell = n;$$

$$\text{or } \quad k = 1,\ m = 2,\ \ell = n+1;$$

$$\text{or } \quad k = 2,\ m = 1,\ n = \ell + 1.$$

We now turn to the Lucas sequence $\{L_n\}$ and prove

Theorem 2.4: If $\dfrac{L_{k+\ell}}{L_k} = \dfrac{L_{m+n}}{L_m}$, with $k, m \ge 1$, $\ell, n \ge 1$, then $k = m$, $\ell = n$

Proof: As in the proof of Theorem 2.2, we observe that, effectively, we have only to show that the assumption $k = 1$, $m \ge 2$ leads to a contradiction. Thus we are given that

$$L_{\ell+1} = \frac{L_{m+n}}{L_m},\ m \ge 2.$$

By Theorem 1.1,

$$\frac{L_{m+n}}{L_m} \in (F_{n+1}, F_{n+2}].$$

But

$$L_{\ell+1} = F_\ell + F_{\ell+2} \in (F_{\ell+2}, F_{\ell+3}].$$

Thus we must have $n = \ell + 1$, so that $L_m L_n = L_{m+n}$. Now it is easy to see that, for all m, n, $L_m L_n = L_{m+n} + (-1)^n L_{m-n}$. Since no Lucas number is zero, we have arrived at the hoped for contradiction.

We use Corollary 2.3 and Theorem 2.4 to obtain interesting results on lcms of Fibonacci and Lucas numbers. We have the well-known classical result

Theorem 2.5: Let $\gcd(a, b) = d$. Then $\gcd(F_a, F_b) = F_d$.

We prove

Theorem 2.6: Let $\operatorname{lcm}(a, b) = \ell$. Then $\operatorname{lcm}(F_a, F_b) = F_\ell \Leftrightarrow a|b$ or $b|a$.

Proof: Certainly, if $a|b$, then $F_a|F_b$, so $\operatorname{lcm}(F_a, F_b) = F_b$ and $b = \ell$. A similar argument applies if $b|a$. Now suppose that $\operatorname{lcm}(F_a, F_b) = F_\ell$. It then follows from Theorem 2.5 that

$$F_a F_b = F_\ell F_d \quad \text{or} \quad \frac{F_a}{F_d} = \frac{F_\ell}{F_b}.$$

Of course, $a \geq d$, $\ell \geq b$. Moreover, $a = d \Leftrightarrow \ell = b$, and this is the case $a|b$. If we are not in this case, then we may apply Corollary 2.3 to infer that

$$\begin{aligned} &a = \ell,\ d = b \\ \text{or }\ &d = 1,\ b = 2,\ a = \ell \\ \text{or }\ &d = 2,\ b = 1,\ a = \ell. \end{aligned}$$

However, the conjunction $d = 1$, $b = 2$, $a = \ell$ is absurd and the conjunction $d = 2$, $b = 1$, $a = \ell$ is even more absurd. Hence, we conclude that $a = \ell$, $d = b$, or $b|a$.

Now there is a result for Lucas numbers corresponding to Theorem 2.5 (see [1] and [2]); thus,

Theorem 2.7: Let $\gcd(a, b) = d$. Then

$$\gcd(L_a, L_b) = \begin{cases} L_d & \text{if } |a|_2 = |b|_2, \\ 2 & \text{if } |a|_2 \neq |b|_2 \text{ and } 3|d, \\ 1 & \text{if } |a|_2 \neq |b|_2 \text{ and } 3\nmid d. \end{cases}$$

Here $|n|_2$ is the 2-valuation of n, i.e., the largest k such that $2^k|n$. Let us say ***a* divides *b* oddly** if $a|b$ and $|a|_2 = |b|_2$, that is, if $a|b$ with odd quotient. Then we prove

Theorem 2.8: Let $\operatorname{lcm}(a, b) = \ell$. Then

$$\operatorname{lcm}(L_a, L_b) = L_\ell \Leftrightarrow a|b \text{ oddly or } b|a \text{ oddly or } a = 1 \text{ or } b = 1.$$

Proof: If $a|b$ oddly, then $L_a|L_b$, so $\operatorname{lcm}(L_a, L_b) = L_b = L_\ell$. A similar argument applies if $b|a$ oddly. Clearly if $a = 1$ or $b = 1$, then $\operatorname{lcm}(L_a, L_b) = L_\ell$. Now suppose conversely that $\operatorname{lcm}(L_a, L_b) = L_\ell$. Obviously we can have $a = 1$ or $b = 1$, so suppose $a, b \geq 2$. Then $L_a|L_\ell$, $L_b|L_\ell$, so, as an easy consequence of Theorem 2.7,

$$|a|_2 = |\ell|_2 = |b|_2.$$

Thus, by Theorem 2.7, $\gcd(L_a, L_b) = L_d$, whence $L_a L_b = L_d L_\ell$ or $L_a / L_d = L_\ell / L_b$. As in the proof of Theorem 2.6, we have $a \geq d$, $\ell \geq b$; and $a = d \Leftrightarrow \ell = b$, this being the case $a|b$ oddly. If we are not in this case, then $a > d$, $\ell > b$ so that, by Theorem 2.4, $a = \ell$, $d = b$, yielding $b|a$ oddly.

Remark: It is intriguing to see that, above, it is the fact that the gcd (of Fibonacci or Lucas numbers) *has* a certain desirable property which ensures that the lcm *cannot have* the corresponding property.

We close by proving a result similar to Theorems 2.6 and 2.8 in the *mixed* case. Here our argument is somewhat *ad hoc* (and could have been used in suitably adapted form to provide an alternative proof of Theorems 2.6 and 2.8). We first quote from [1] and [2].

Theorem 2.9: Let $\gcd(a, b) = d$. Then

$$\gcd(F_a, L_b) = \begin{cases} L_d & \text{if } |a|_2 > |b|_2, \\ 2 & \text{if } |a|_2 \leq |b|_2 \text{ and } 3|d, \\ 1 & \text{if } |a|_2 \leq |b|_2 \text{ and } 3\nmid d. \end{cases}$$

Now if $|a|_2 > |b|_2$ and $\ell = \operatorname{lcm}(a, b)$, then $F_a | F_\ell$, $L_b | F_\ell$ (since $2b|\ell$). Thus we may raise the question as to whether $\operatorname{lcm}(F_a, L_b) = F_\ell$. We prove

Theorem 2.10: Let $|a|_2 > |b|_2$. Then $\operatorname{lcm}(F_a, L_b) = F_\ell \Leftrightarrow b|a$.

Proof: If $b|a$, then $2b|a$, so $L_b | F_{2b} | F_a$ and $\operatorname{lcm}(F_a, L_b) = F_a = F_\ell$. Suppose conversely that $\operatorname{lcm}(F_a, L_b) = F_\ell$ and $b \nmid a$, that is, $b \neq d$, $a \neq \ell$. Then, by Theorem 2.9, $F_a L_b = F_\ell L_d$, so

$$L_b / L_d = F_\ell / F_a.$$

Suppose $d \geq 2$. Then, by Theorem 1.1,

$$\frac{L_b}{L_d} \in (F_{b-d+1}, F_{b-d+2}] \quad \text{and} \quad \frac{F_\ell}{F_a} \in (F_{\ell-a+1}, F_{\ell-a+2}].$$

Thus we conclude that $b - d = \ell - a$. This, however, is impossible, since it implies $a = d$ or $b = d$; and $a = d$ is excluded because $a \nmid b$.

Suppose, finally, that $d = 1$. Then

$$L_b = \frac{F_\ell}{F_a} \in (F_{\ell-a+1}, F_{\ell-a+2}].$$

But $L_b \in (F_{b+1}, F_{b+2}]$, which implies $\ell - a = b$. This, in turn, implies $ab = a + b$ with a, b mutually prime, which is plainly absurd. This final contradiction establishes the theorem.

REFERENCES

1. Peter Hilton & Jean Pedersen. "Fibonacci and Lucas Numbers in Teaching and Research." *Journées Mathématiques & Informatique* **3** (1991-1992):36-57.
2. Wayne McDaniel. "The gcd in Lucas Sequences and Lehmer Number Sequences." *The Fibonacci Quarterly* **29.1** (1991):24-29.

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