CONCERNING THE RECURSIVE SEQUENCE $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{\alpha_i}$

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(Submitted October 1993)

1. MAIN RESULT

In [1] H. T. Freitag has raised a conjecture that for the sequence $\{A_n\}$, defined by $A_{n+2} = \sqrt{A_{n+1}} + \sqrt{A_n}$ for all $n \geq 1$, $\lim_{n\to\infty} A_n = 4$ regardless of the choice of $A_1, A_2 > 0$. In this note we will give a positive answer to this conjecture by proving the following more general theorem.

Theorem 1: If $-1 < \alpha_i < 1$, $1 \leq i \leq k$ and $A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}^{\alpha_i}$, $n \geq 1$, then

$$\lim_{n\to\infty} A_n = L,$$

the unique root of the equation $\sum_{i=1}^{k} a_i x^{\alpha_i - 1} - 1 = 0$ in the interval $(0, \infty)$, regardless of the choice of $A_i > 0$, $1 \leq i \leq k$, where $a_i \geq 0$, $1 \leq i \leq k$, and $\sum_{i=1}^{k} a_i > 0$.

In particular, if $k = 2$, $a_i = a_2 = 1$, and $\alpha_i = \alpha_2 = 1/2$, we have

$$\lim_{n\to\infty} A_n = 4.$$

This coincides with Freitag's conjecture.

Proof: Let $A_n = Lx_n$. Then

$$x_{n+k} = \sum_{i=1}^{k} \beta_i x_{n+i-1}^{\alpha_i},$$

where $\beta_i = a_i L^{\alpha_i - 1}$, and therefore

$$\sum_{i=1}^{k} \beta_i = 1. \tag{1}$$

Obviously, we only need to prove that

$$\lim_{n\to\infty} x_n = 1. \tag{2}$$

To this end, set $M = \max\{x_i, x_i^{-1};\ 1 \leq i \leq k\}$ and $\alpha = \max\{|\alpha_1|, \ldots, |\alpha_k|\}$. It is obvious that $M \geq 1$, $0 \leq \alpha < 1$, and

$$M \geq x_i \geq M^{-1},\ 1 \leq i \leq k. \tag{3}$$

We will use induction to prove that

$$M^{\alpha^n} \geq x_{kn+i} \geq M^{-\alpha^n},\ 1 \leq i \leq k, \tag{4}$$

* The work of this author is supported by the Alexander von Humboldt Foundation and the Natural Science Foundation of China.

holds for all $n \geq 0$. In fact, from (3), (4) holds when $n = 0$. We assume that (4) holds if $n \leq \ell - 1$. For $n = \ell$, from the induction assumption and the definition of M, it follows that

$$M^{\alpha^\ell} \geq M^{|\alpha_i|\alpha^{\ell-1}}, \quad 1 \leq i \leq k, \tag{5}$$

and

$$M^{-|\alpha_i|\alpha^{\ell-1}} \leq x_{(\ell-1)k+i}^{\alpha_i} \leq M^{|\alpha_i|\alpha^{\ell-1}}, \quad 1 \leq i \leq k. \tag{6}$$

Therefore, from (5) and (6), we have

$$x_{k\ell+1} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i}^{\alpha_i} \leq \sum_{i=1}^{k} \beta_i M^{|\alpha_i|\alpha^{\ell-1}} \leq M^{\alpha^\ell},$$

and, furthermore, we have

$$x_{k\ell+2} = \sum_{i=1}^{k} \beta_i x_{(\ell-1)k+i+1}^{\alpha_i} \leq \sum_{i=1}^{k-1} \beta_i M^{|\alpha_i|\alpha^{\ell-1}} + \beta_k M^{|\alpha_k|\alpha^\ell} \leq M^{\alpha^\ell}.$$

In the last step we have used the fact that $M^{|\alpha_k|\alpha^\ell} \leq M^{\alpha^\ell}$. Similarly, the left-hand inequality of (4) holds for $n = \ell$ and other indices i, $3 \leq i \leq k$. The right-hand inequality of (4) can be justified in a similar way. Noting that $0 \leq \alpha < 1$, we obtain

$$\lim_{n\to\infty} M^{-\alpha^n} = \lim_{n\to\infty} M^{\alpha^n} = 1.$$

By (4), this implies that (2) holds. □

Corollary 1: If $-1 < \alpha_1 = \cdots \alpha_k = \alpha < 1$ and $a_1 = \cdots a_k = 1$, then

$$\lim_{n\to\infty} A_n = k^{\frac{1}{1-\alpha}},$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^\infty$ is as defined in Theorem 1.

Corollary 2: If $-1 < \alpha_i < 1$, $a_i \geq 0$, and $\sum_{i=1}^{k} a_i = 1$, then

$$\lim_{n\to\infty} A_n = 1,$$

independent of the choice of $A_1, A_2, \ldots, A_k > 0$, where $\{A_n\}_1^\infty$ is also as defined in Theorem 1. Corollary 2 follows from the fact that $L = 1$ is the only root of the equation $\sum_{i=1}^{k} a_i x^{\alpha_i - 1} - 1 = 0$ in the interval $(0, \infty)$.

2. FURTHER RESULTS

In this section we consider a *linear* recursive sequence, that is, when we choose $\alpha_i = 1$, $1 \leq i \leq k$, in the recursive sequence considered above.

Theorem 2: Let the complex sequence $\{A_n\}_1^\infty$ satisfy

$$A_{n+k} = \sum_{i=1}^{k} a_i A_{n+i-1}.$$

Then, if $a_i>0$, $1\le i\le k$, and $\sum_{i=1}^{k}a_i=1$, the sequence $\{A_n\}_1^\infty$ converges to a limit which depends on the values of A_i, $1\le i\le k$.

Proof: We will prove that $x=1$ is a single root of the eigenpolynomial,

$$p(x):=x^k-\sum_{i=1}^{k}a_ix^{i-1}=0, \tag{7}$$

of the recursive sequence

$$A_{n+k}=\sum_{i=1}^{k}a_iA_{n+i-1},$$

and the moduli of all other roots of (7) are less than 1.

In fact, since $\sum_{i=1}^{k}a_i=1$, we have $p(1)=0$. This means that $x=1$ is a root of $p(x)$. From

$$p'(1)=k-\sum_{i=1}^{k}(i-1)a_i\ge 1,$$

it follows that $x=1$ is a single root of $p(x)$. On the other hand, for $x=re^{i\theta}$, $r\ge 1$, and $0\le\theta<2\pi$, we have

$$\left|p(re^{i\theta})\right|\ge r^k-\left|\sum_{j=1}^{k}a_jr^{j-1}e^{(j-1)\theta i}\right|\ge\left(r-\sum_{i=1}^{k}a_i\right)r^{k-1}\ge 0.$$

It is easy to see that the above inequalities become equalities if and only if $r=1$ and $\theta=0$. Therefore, if $x=x_0$ is a zero of $p(x)$, then $|x_0|\le 1$ and $x_0=1$ when $|x_0|=1$. Set

$$p(x)=(x-1)(x-x_1)^{r_1}\cdots(x-x_m)^{r_m}, \tag{8}$$

where $1+r_1+\cdots r_m=k, |x_j|<1$, $1\le j\le m$, and $x_i\ne x_j$ when $i\ne j$. It is well known that $\{A_n\}_1^\infty$ has the general solution

$$A_n=c+\sum_{i=1}^{m}\sum_{j=0}^{r_i-1}c_{i,j}n^jx_i^n. \tag{9}$$

From (9), we deduce that

$$\lim_{n\to\infty}A_n=c.$$

The value of c depends on the choice of A_j, $1\le j\le k$. This completes the proof of Theorem 2. □

Note: Theorem 1 and Theorem 2 can be generalized easily to discuss sequences of functions. To state this precisely, we have

Theorem 3: Let $a_i=a_i(x)$ and $\alpha_i=\alpha_i(x)$, $1\le i\le k$, be functions defined on a point set $I\subset R^m$, a Euclidean space, and let the function sequence $\{A_n(x)\}_1^\infty$ be defined as

$$A_{n+k}(x)=\sum_{i=1}^{k}a_iA_{n+i-1}^{\alpha_i}(x),\ n\ge 1.$$

Then we have:

(1) If $a_i(x) \geq 0$ and $-1 < \alpha_i(x) < 1$ hold for an $x \in I$, $\{A_n(x)\}_1^\infty$ converges at the point x to $L = L(x)$, the unique root of $\sum_{i=1}^{k} a_i y^{\alpha_i - 1} = 1$ if $a_i(x)$, $1 \leq i \leq k$, are not all zeros and the sequence converges pointwise to zero if $a_i(x) = 0$ for all i, $1 \leq i \leq k$, regardless of the choice of $A_i(x) > 0$, $1 \leq i \leq k$;

(2) If $a_i(x) \geq 0$, $\sum_{i=1}^{k} a_i(x) = 1$, and $\alpha_i(x) = 1$, $1 \leq i \leq k$, hold for an $x \in I$, $\{A_n(x)\}_1^\infty$ converges at the point x. In particular, for case *(1)*, $\{A_n(x)\}_1^\infty$ converges uniformly if there are constants α, $0 \leq \alpha < 1$, $a > 0$, and M such that $|\alpha_i(x)| \leq \alpha$, $1 \leq i \leq k$, $0 < \sum_{i=1}^{k} a_i(x) \leq a$, $x \in I$, and $\sup_{x \in I} \{A_i(x), A_i^{-1}(x) | 1 \leq i \leq k\} \leq M$ hold, respectively.

ACKNOWLEDGMENT

The author would like to give his hearty thanks to the referee for valuable suggestions which improved the presentation of this note.

REFERENCE

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AMS Classification Numbers: 11B37, 11B39