ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-499 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Given n a natural number, n is a *Lucas pseudoprime* (LPP) if it is composite and satisfies the following congruence:

$$L_n \equiv 1 \pmod{n}. \tag{1}$$

If $\gcd(n, 10) = 1$, the *Jacobi symbol* $(5/n) = \varepsilon_n$ is given by the following:

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } n \equiv \pm 3 \pmod{10}. \end{cases}$$

Given $\gcd(n, 10) = 1$, n is a *Fibonacci pseudoprime* (FPP) if it is composite and satisfies the following congruence:

$$F_{n-\varepsilon_n} \equiv 0 \pmod{n}. \tag{2}$$

Define the following sequences for $e = 1, 2, \ldots$:

$$u = u_e = F_{3^{e+1}} / F_{3^e}; \tag{3}$$

$$v = v_e = L_{3^{e+1}} / L_{3^e}; \tag{4}$$

$$w = w_e = F_{2\cdot 3^{e+1}} / F_{2\cdot 3^e} = uv. \tag{5}$$

Prove the following for all $e \ge 1$:

(i) u is a FPP and a LPP, provided it is composite;

(ii) same statement for v;

(iii) w is a FPP but not a LPP.

H-500 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \ge 2$. Show that for all complex numbers x and all nonnegative integers n,

$$\sum_{k=0}^{[n/2]}\binom{2n+2}{n-2k}F_{2k+1}(x)=x^nF_{n+1}(4/x), \tag{1}$$

where [ ] denotes the greatest integer function.

As special cases of (1), obtain the following identities:

$$\sum_{k=0}^{[n/2]}\binom{2n+2}{n-2k}F_{2k+1}=\frac{1}{2}F_{3n+3}; \tag{2}$$

$$\sum_{k=0}^{[n/2]}\binom{2n+2}{n-2k}F_{6k+3}=2^{2n+1}F_{n+1}; \tag{3}$$

$$\sum_{k=0}^{[n/2]}\binom{2n+2}{n-2k}L_{4k+2}=\frac{1}{2}(5^{n+1}-(-1)^{n+1}). \tag{4}$$

H-501 *Proposed by Paul S. Bruckman, Edmonds, WA*

Define the following sequences for $e=1,2,\ldots$:

(i) $u=u_e=F_{5^{e+1}}/5F_{5^e}$;

(ii) $v=v_e=L_{5^{e+1}}/L_{5^e}$;

(iii) $w=w_e=F_{2\cdot5^{e+1}}/5F_{2\cdot5^e}=uv$.

Prove the following:

(a) If u is composite, it is both a Fibonacci pseudoprime (FPP) and a Lucas pseudoprime (LPP); see Problems H-496 and H-498 for definitions of FPP's and LPP's.

(b) Same problem for v.

(c) Show that w is a FPP but not a LPP.

H-502 *Proposed by Zdzislaw W. Trzaska, Warsaw, Poland*

Given two sequences of polynomials in complex variable $z\in C$ defined recursively as

$$T_k(z)=\sum_{m=0}^{k}a_{km}z^m,\ k=0,1,2,\ldots, \tag{1}$$

with $T_0(z)=1$ and $T_1(z)=(1+z)T_0$, and

$$P_k(z)=\sum_{m=0}^{k}b_{km}z^m,\ k=0,1,2,\ldots, \tag{2}$$

with $P_0(z)=0$ and $P_1(z)=1$.

Prove that for all $z\in C$ and $k=0,1,2,\ldots$, the equality

$$P_k(z)T_{k-1}(z)-T_k(z)P_{k-1}(z)=1 \tag{3}$$

holds.

SOLUTIONS

Eventually

H-485 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 32, no. 1, February 1994)

If x is an unspecified large positive real number, obtain an asymptotic evaluation for the sum $S(x)$, where

$$S(x) = \sum_{p \le x} (-1)^{Z(p)}; \tag{1}$$

here, the p's are prime and $Z(p)$ is the Fibonacci entry-point of p (the smallest positive n such that $p \mid F_n$).

Solution by the proposer

Let $\mathscr{L} = \{L_n\}_{n\ge 0}$ denote the Lucas sequence. It is well known that $Z(p)$ is even iff $p \in \rho(\mathscr{L})$, where $\rho(\mathscr{L})$ denotes the set of primes p such that p divides an element of $\mathscr{L}$. Let $\pi_{\mathscr{L}}(x)$ denote the number of primes $p \in \rho(\mathscr{L})$ with $p \le x$; also, $\pi(x)$ denotes the number of $p \le x$. The density of $\rho(\mathscr{L})$ is defined as $\lim_{x\to\infty} \pi_{\mathscr{L}}(x)/\pi(x) \equiv \theta_{\mathscr{L}}$, assuming such a limit exists.

In 1985, Lagarias showed [1], among other things, that $\theta_{\mathscr{L}} = 2/3$. We see that this result is equivalent to the following:

$$\lim_{x\to\infty} A(x)/\pi(x) = \theta_{\mathscr{L}} = 2/3, \tag{2}$$

where

$$A(x) \equiv \sum_{\substack{p\le x\\ Z(p)\text{ even}}} 1; \text{ also, } B(x) \equiv \sum_{\substack{p\le x\\ Z(p)\text{ odd}}} 1. \tag{3}$$

Also note that $A(x) - B(x) = S(x)$ and $A(x) + B(x) = \pi(x)$. Moreover, we recall the famous Prime Number Theorem, namely,

$$\pi(x) \sim \frac{x}{\log x} \quad (\text{as x} \to \infty). \tag{4}$$

Consequently, we see that

$$A(x) \sim \frac{2x}{3\log x}, \quad B(x) \sim \frac{x}{3\log x}, \quad \text{and} \quad S(x) \sim B(x),$$

or:

$$S(x) \sim \frac{x}{3\log x}. \tag{5}$$

Reference

1. J. C. Lagarias. "The Set of Primes Dividing the Lucas Numbers Has Density 2/3." *Pacific J. Math.* **118** (1985):19-23.

Long Range PI

H-486 *Proposed by Piero Filipponi, Rome, Italy (Vol. 32, no. 2, May 1994)*

Let the terms of the sequence $\{Q_k\}$ be defined by the second-order recurrence relation $Q_k = 2Q_{k-1} + Q_{k-2}$ with initial conditions $Q_0 = Q_1 = 1$. Find restrictions on the positive integers n and m for

$$T(n, m) = \sum_{k=1}^{\infty} \frac{k^2 n^k Q_k}{m^k}$$

to converge, and, under these restrictions, evaluate this sum. Moreover, find the set of all couples (n_i, m_i) for which $T(n_i, m_i)$ is an integer.

Solution by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC

The ratio test shows that the series will converge if

$$\frac{n}{m} < \sqrt{2} - 1 \approx .4142. \text{ Thus, } \left| \frac{n}{m}(1 \pm \sqrt{2}) \right| < 1.$$

Since $Q_k = \frac{1}{2}[(1+\sqrt{2})^k + (1-\sqrt{2})^k]$, it follows from the summation formula

$$\sum_{k=1}^{\infty} k^2 x^k = \frac{x(1+x)}{(1-x)^3}, \; |x| < 1,$$

that

$$T = \sum_{k=1}^{\infty} k^2 \left(\frac{n}{m}\right)^k Q_k = \frac{1}{2}\left(\sum_{k=1}^{\infty} k^2 \left[\frac{n}{m}(1+\sqrt{2})\right]^k + \sum_{k=1}^{\infty} k^2 \left[\frac{n}{m}(1-\sqrt{2})\right]^k \right)$$

simplifies to

$$T(n, m) = \frac{nm(m^2 + n^2)(m^2 + 6mn - n^2)}{(m^2 - 2mn - n^2)}.$$

The only values of (n, m) for which T will be integral are those satisfying $m^2 - 2mn - n^2 = 1$ or $(n-m)^2 = 1 + 2n^2$. The equation

$$x^2 - 2n^2 = 1$$

is a Pell equation with the fundamental solution $x_1 = 3$ and $n_1 = 2$. Thus, the solution set $\{(x_k, n_k)\}$ is generated from

$$x_k + y_k\sqrt{2} = (3 + 2\sqrt{2})^k.$$

Since $m = x + n$ all pairs (n, m) leading to the integral values of $T(n, m)$ are determined. The first four are $(2, 5)$, $(12, 29)$, $(70, 169)$, and $(408, 985)$. The first three integral values of T are 23490, 954642300, and 37463036986830.

Also solved by P. Bruckman, H.-J. Seiffert, and the proposer (partial solution).

Nice Couples

H-487 ***Proposed by Stanley Rabinowitz, MathPro Press, Westford, MA (Vol. 32, no. 2, May 1994)***

Suppose H_n satisfies a second-order linear recurrence with constant coefficients. Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, r$, be integer constants and let $f(x_0, x_1, x_2, \ldots, x_r)$ be a polynomial with integer coefficients. If the expression

$$f((-1)^n, H_{a_1n+b_1}, H_{a_2n+b_2}, \ldots, H_{a_rn+b_r})$$

vanishes for all integers $n > N$, prove that the expression vanishes for all integral n.

[As a special case, if an identity involving Fibonacci and Lucas numbers is true for all positive subscripts, then it must also be true for all negative subscripts as well.]

Solution by Paul S. Bruckman, Edmonds, WA

Let $\underline{H}_n = \{(-1)^n, H_{a_1n+b_1}, H_{a_2n+b_2}, \ldots, H_{a_rn+b_r}\}$ and $f_n \equiv f(\underline{H}_n)$. Also, let Π_n denote any product of the form $(-1)^{ne_0}H^{e_1}_{a_1n+b_1} \cdots H^{e_r}_{a_rn+b_r}$, $e_i \geq 0$ and integers. Since H_n has a nullifying (i.e., characteristic) polynomial satisfying the recurrence relation $P(E)(H_n) = 0$ (here E is the unit "right-shift" operator, with n the operand, and P is a polynomial with constant coefficients, of second degree), it follows that $H_{a_in+b_i}$ also has a nullifying polynomial; then so does $H^{e_i}_{a_in+b_i}$, where the integers e_i are nonnegative. The same is true for $(-1)^{ne_0}$, for which the nullifying polynomial is $E-(-1)^{e_0}$. Then any product Π_n has a nullifying polynomial; since f_n is a sum of products of the form Π_n, it follows that f_n itself has a nullifying polynomial, say $G(x)$. Thus, $G(E)(f_n) = 0$ *for all n*. We may suppose that $G(x) = \sum_{j=m}^{M} c_jx^j$, where $M \geq m \geq 0$, $c_m \neq 0$, $c_M \neq 0$. We consider two possibilities:

(a) $m = M$—then $G(E)(f_n) = c_mf_{m+n} = 0$ for all n, which implies $f_n = 0$ for all n;

(b) $M > m \geq 0$. Then $G(E)(f_{N-m}) = \sum_{j=m}^{M} c_jf_{N-m+j} = c_mf_N = 0$, since $f_{N+1} = f_{N+2} = \cdots = 0$, by hypothesis. Thus, $f_N = 0$.

We may repeat the process (i.e., setting $n = N-1-m$, $N-2-m$, etc.), and conclude that $f_n = 0$ for all n. Q.E.D.

Pseudo Nim

H-488 ***Proposed by Paul S. Bruckman, Edmonds, WA (Vol. 32, no. 4, August 1994)***

The *Fibonacci pseudoprimes* (FPP's) are those composite integers n with $\gcd(n, 10) = 1$ and satisfying the following congruence:

$$F_{n-\varepsilon_n} \equiv 0 \pmod{n}, \tag{i}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10}, \\ -1 & \text{if } n \equiv \pm 3 \pmod{10}. \end{cases}$$

$$\left[\text{Thus, } \varepsilon_n = (\tfrac{5}{n}), \text{ a Jacobi symbol.}\right]$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2_p}$ is a FPP if u is composite.

The *Lucas pseudoprimes* (LPP's) are those composite positive integers n satisfying the following congruence:

$$L_n \equiv 1 \pmod{n}. \tag{ii}$$

Given a prime $p > 5$, prove that $u = \frac{1}{3}L_{2_p}$ is a LPP if u is composite.

Solution by Norbert Jensen, Kiel, Germany

Step 0: 3 divides L_{2p}; hence, u is always an integer.

Proof: Consider the Lucas numbers modulo 3: $L_0 = 2$, $L_1 = 1$, $L_2 \equiv 0$, $L_3 \equiv 1$, $L_4 \equiv 1$, $L_5 \equiv 2$, $L_6 \equiv 0$, $L_7 \equiv 2$, $L_8 \equiv 2 = L_0$, $L_9 \equiv 1 = L_1 \pmod 3$. Hence, $(L_n)_{n \in \mathbb{N}_0}$ has period length 8 modulo 3 and $L_n \equiv 0 \pmod 3$ if and only if $n \equiv 2$ or $n \equiv 6 \pmod 8$. But as $p \equiv 1$ or $\equiv 3 \pmod 4$, it is clear that $2p \equiv 2$ or $2p \equiv 6 \pmod 8$. Q.E.D. (Step 0)

Suppose that u is composite. We have to show that u is a FPP and a LPP.

Step 1: We show that $u \equiv 1 \pmod{10}$. Hence, $\gcd(u, 10) = 1$.

Proof: Consider the residues of $(L_n)_{n \in \mathbb{N}_0}$ mod 10: $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 \equiv -3$, $L_5 \equiv 1$, $L_6 \equiv -2$, $L_7 \equiv -1$, $L_8 \equiv -3$, $L_9 \equiv -4$, $L_{10} \equiv 3$, $L_{11} \equiv -1$, $L_{12} \equiv 2 = L_0$, $L_{13} \equiv 1 = L_1$. Hence, the sequence $(L_n)_{n \in \mathbb{N}_0}$ has period length 12 mod 10. As p is either $\equiv 1$ or $\equiv -1$ mod 6, it follows that $2p$ is either $\equiv 2$ or $\equiv -2$ mod 12. Hence, $L_{2p} \equiv L_2 = 3$ or $L_{2p} \equiv L_{10} \equiv 3$ mod 10. Cancelling 3 in the above congruences shows that $u \equiv 1 \pmod{10}$. Q.E.D. (Step 1)

In particular, we have $\varepsilon_u = 1$. So to prove that u is a FPP and a LPP, we have to demonstrate that $F_{u-1} \equiv 0$, $L_u \equiv 1 \pmod u$.

Step 2: We show that $L_{8p} \equiv 2$, $L_{8p+1} \equiv 1$, $F_{8p} \equiv 0$, $F_{8p+1} \equiv 1 \pmod u$. Hence, $8p$ is a common period of the Lucas and the Fibonacci sequence modulo u.

(Actually, it can be shown that—in terms of algebraic number theory—the order of α modulo the ideal $u\mathbb{Z}[\alpha]$ in $\mathbb{Z}[\alpha]$ is $8p$.)

Proof: From the definition of u, it follows that $\alpha^{2p} + \beta^{2p} = L_{2p} = 3u$ or $\alpha^{2p} = -\beta^{2p} + 3u$. By multiplication with $(1/\beta)^{2p} = (-\alpha)^{2p} = \alpha^{2p}$, we obtain $\alpha^{4p} = -1 + 3u\alpha^{2p}$. Squaring both sides, we arrive at

$$\alpha^{8p} = 1 - 6u\alpha^{2p} + 9u^2\alpha^{4p}. \tag{2.1}$$

Exchanging α and β in these operations leads to

$$\beta^{8p} = 1 - 6u\beta^{2p} + 9u^2\beta^{4p}. \tag{2.2}$$

Multiplying (2.1) and (2.2) by α and β, respectively, we obtain

$$\alpha^{8p+1} = \alpha - 6u\alpha^{2p+1} + 9u^2\alpha^{4p+1}. \tag{2.3}$$

$$\beta^{8p+1} = \beta - 6u\beta^{2p+1} + 9u^2\beta^{4p+1}. \tag{2.4}$$

Summing up (2.1), (2.2) and (2.3), (2.4) gives $L_{8p} \equiv 2 \pmod{u}$, $L_{8p+1} \equiv 1 \pmod{u}$. Now, subtracting (2.2) from (2.1) and (2.4) from (2.3) and multiplying with $\alpha - \beta = \sqrt{5}$ gives $5F_{8p} \equiv 0$ (mod u), $5F_{8p+1} \equiv 5 \pmod{u}$. By Step 1, 5 does not divide u; hence, we cancel 5 in the above congruences. Thus, $F_{8p} \equiv 0$, $F_{8p+1} \equiv 1 \pmod{u}$. Q.E.D. (Step 2)

It remains to show that $u \equiv 1 \pmod{8p}$.

Splitting up into congruences modulo prime powers, we obtain the following results (i.e., Steps 3 and 4).

Step 3: We show that $L_{2p} \equiv 3 \pmod{8}$.

Proof: First, we determine the period length of $(L_n)_{n\in\mathbb{N}_0}$ mod 8. We have $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 \equiv -1$, $L_5 \equiv 3$, $L_6 \equiv 2$, $L_7 \equiv -3$, $L_8 \equiv -1$, $L_9 \equiv 4$, $L_{10} \equiv 3$, $L_{11} \equiv -1$, $L_{12} \equiv 2 = L_0$, $L_{13} \equiv 1 = L_1 \pmod{8}$. Hence, 12 is the period length of $(L_n)_{n\in\mathbb{N}_0}$. Since $p > 5$, we just have to consider the following two cases:

Case 1: $p \equiv 1 \pmod{6}$. Then $2p \equiv 2 \pmod{12}$ and $L_{2p} \equiv L_2 = 3 \pmod{8}$.

Case 2: $p \equiv -1 \pmod{6}$. Then $2p \equiv 10 \pmod{12}$ and $L_{2p} \equiv L_{10} = 3 \pmod{8}$.

Q.E.D. (Step 3)

Step 4: We show that $L_{2p} \equiv 3 \pmod{p}$.

Proof: We need the following two facts:

$$L_{2p} = 2^{1-2p} \sum_{\substack{j=0 \\ j\equiv 0 \,(\mathrm{mod}\, 2)}}^{2p} \binom{2p}{j} 5^{j/2}; \tag{4.1}$$

$$\binom{2p}{j} \equiv 0 \pmod{p} \text{ if either } o < j < p \text{ or } p < j < 2p. \tag{4.2}$$

From these facts, it follows (using Fermat's theorem) that

$$4 \cdot L_{2p} \equiv 2^{2p} L_{2p} \equiv 2 \cdot (1 + 5^p) \equiv 2 \cdot 6 \pmod{p}.$$

Since p and 4 are coprime, we can cancel 4 on both sides of the congruence; whence the assertion follows. Q.E.D. (Step 4)

Step 5: Using Steps 3 and 4, we obtain $L_{2p} \equiv 3 \pmod{8p}$. Now, by Step 0, and since 3 and $8p$ are coprime, it follows that $u \equiv 1 \pmod{8p}$.

Step 6: Applying Steps 2 and 5, we see that $L_u \equiv L_1 = 1 \pmod{u}$ and $F_{u-1} \equiv 0 \pmod{u}$. Q.E.D.

Also solved by H.-J. Seiffert and the proposer.

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