VISUALIZING GOLDEN RATIO SUMS WITH TILING PATTERNS

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Several sums involving the Golden Ratio $\Phi = (1+\sqrt{5})/2$ can be illustrated by tiling either squares or golden rectangles with squares, rectangles, gnomons, or other shapes formed from rectangles. This visually-pleasing approach complements an early paper, "Fibonacci Numbers and Geometry," by Brother Alfred Brousseau [1].

The basic golden rectangle, with ratio length to width Φ, is the basis for all figures that follow. In Figure 1, the length is 1 and the width is $1/\Phi$.

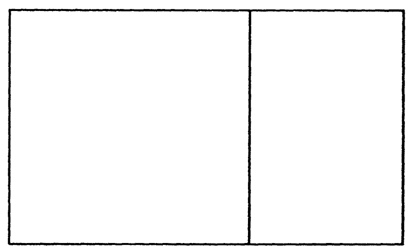

FIGURE 1: The Golden Rectangle

Divide the sides of a square and a golden rectangle in powers of $1/\Phi$ to form the templates of Figure 2.

In Figure 3 a square of side 1 tiled with golden rectangles shows that

$$1/\Phi + 1/\Phi^3 + 1/\Phi^5 + \cdots + 1/\Phi^{2n-1} + \cdots = 1$$

while a golden rectangle of length 1 tiled with squares (Figure 4) shows that

$$1/\Phi^2 + 1/\Phi^4 + 1/\Phi^6 + \cdots + 1/\Phi^{2n} + \cdots = 1/\Phi.$$

Divide a square of side Φ into powers of $1/\Phi$ and tile the rectangles that lie on falling diagonals to form Figure 5. Then each successive diagonal has n rectangles each of area $1/\Phi^{n+1}$, so that

$$1/\Phi^2 + 2/\Phi^3 + 3/\Phi^4 + \cdots + n/\Phi^{n+1} + \cdots = \Phi^2.$$

Also, the length of each side is $1/\Phi + 1/\Phi^2 + 1/\Phi^3 + \cdots + 1/\Phi^n + \cdots = \Phi$.

In Figure 6 we again divide a square of side Φ into powers of $1/\Phi$ and tile with L-shaped tiles, each formed from two rectangles having area $1/\Phi^{2n-1}$ There are F_n L-shaped tiles, each of area $2/\Phi^{2n-1}$, so that

$$1/\Phi + 1/\Phi^3 + 2/\Phi^5 + \cdots + F_n/\Phi^{2n-1} + \cdots = \Phi^2/2,$$

where F_n is the n^{th} Fibonacci number.

Figure 7 uses gnomons as tiles, where the largest has area $1/\Phi$ and the n^{th} gnomon has area $1/\Phi^{2n-1}$, making a visualization of the formula

$$1/\Phi + 1/\Phi^3 + 1/\Phi^5 + \cdots + 1/\Phi^{2n-1} + \cdots = 1.$$

Figure 8 is similar to Figure 5, but the tiling distinguishes squares, rectangles of area $1/\Phi^{2n}$, and rectangles of area $1/\Phi^{2n+1}$. The $(2n-1)^{\text{st}}$ diagonal contains one square of area $1/\Phi^{2n}$ and $(2n-2)$ rectangles each of area $1/\Phi^{2n}$, while the $(2n)^{\text{th}}$ diagonal contains $2n$ rectangles each of area $1/\Phi^{2n+1}$. Figure 8 provides a visualization of the sums:

$$1/\Phi^2 + 1/\Phi^4 + 1/\Phi^6 + \cdots + 1/\Phi^{2n} + \cdots = 1/\Phi;$$
$$1/\Phi^4 + 2/\Phi^6 + 3/\Phi^8 + \cdots + n/\Phi^{2n+2} + \cdots = 1/\Phi^2;$$
$$1/\Phi^3 + 2/\Phi^5 + 3/\Phi^7 + \cdots + n/\Phi^{2n+1} + \cdots = 1/\Phi.$$

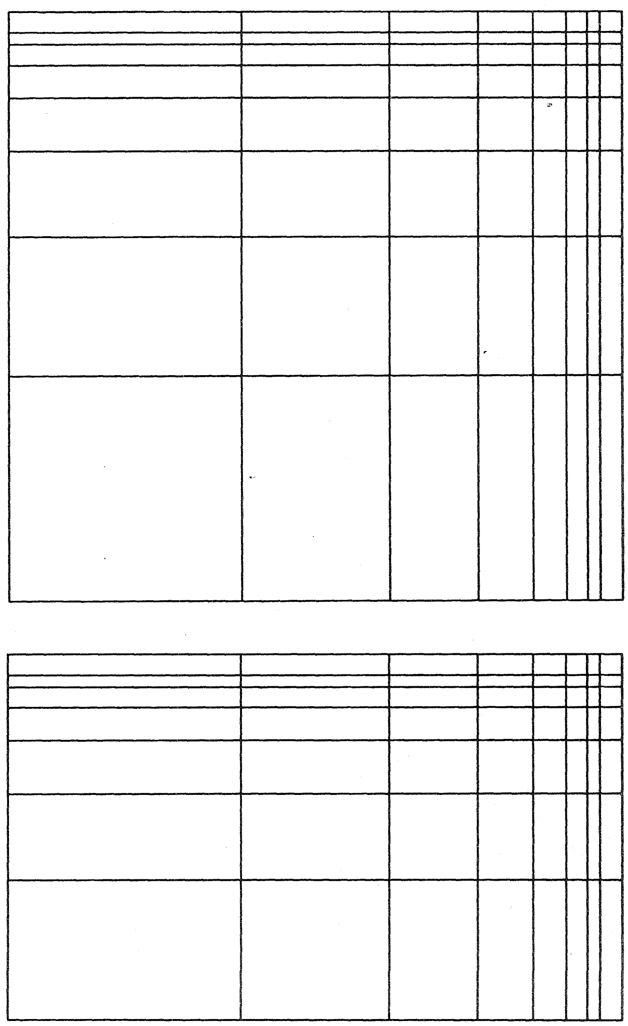

FIGURE 2: Templates for Visualizing Fibonacci and Golden Ratio Summation Formulas with Tiling Patterns

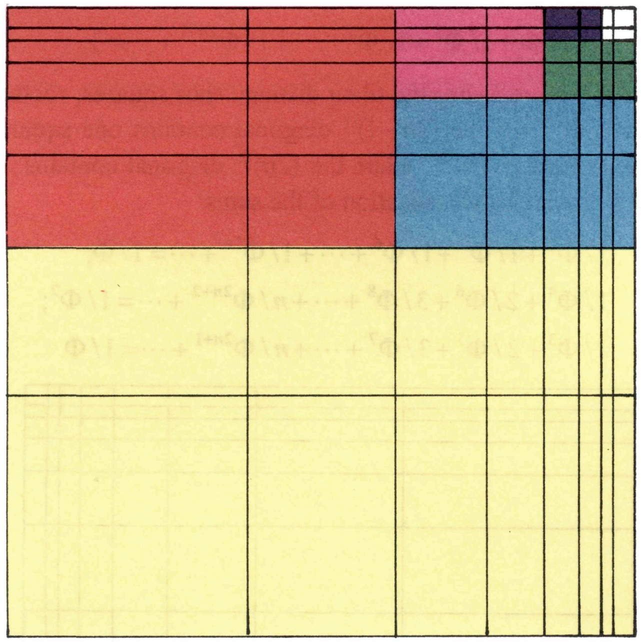

FIGURE 3: $1/\Phi + 1/\Phi^3 + 1/\Phi^5 + \cdots + 1/\Phi^{2n-1} + \cdots = 1$

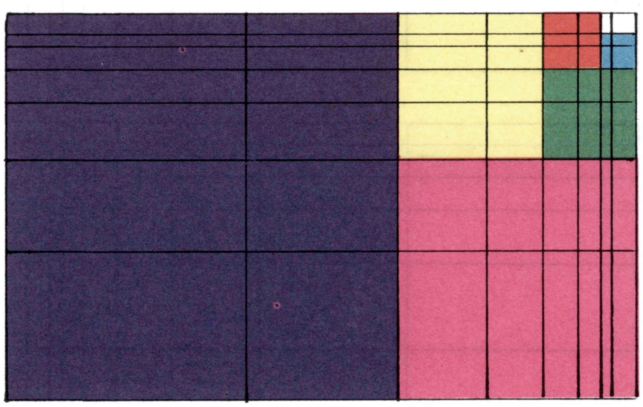

FIGURE 4: $1/\Phi^2 + 1/\Phi^4 + 1/\Phi^6 + \cdots + 1/\Phi^{2n} + \cdots = 1/\Phi$

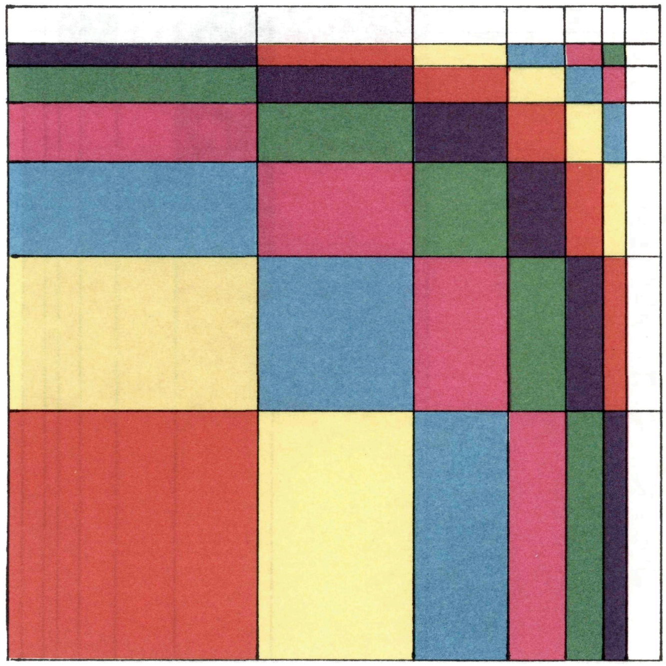

FIGURE 5: $1/\Phi^2 + 2/\Phi^3 + 3/\Phi^4 + \cdots + n/\Phi^{n+1} + \cdots = \Phi^2$

$1/\Phi + 1/\Phi^2 + 1/\Phi^3 + \cdots + 1/\Phi^n + \cdots = \Phi$

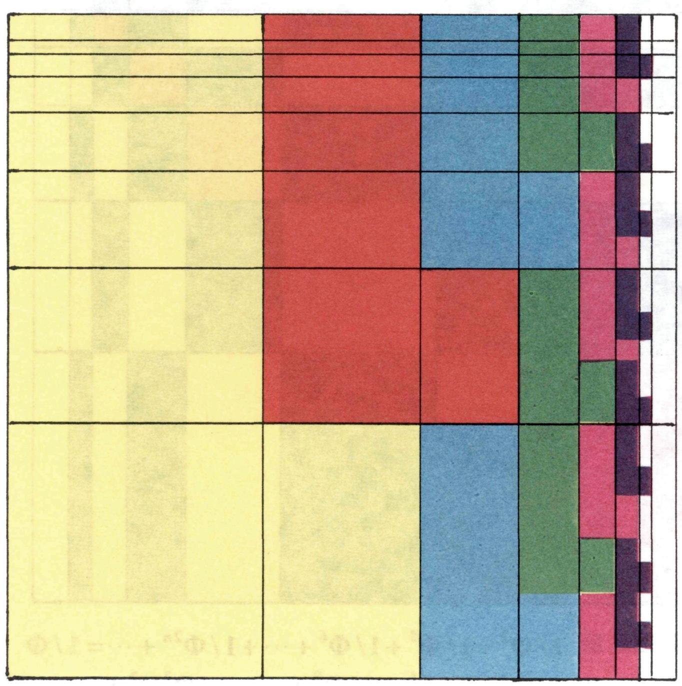

FIGURE 6: $1/\Phi + 1/\Phi^3 + 2/\Phi^5 + \cdots + F_n/\Phi^{2n-1} + \cdots = \Phi^2/2$

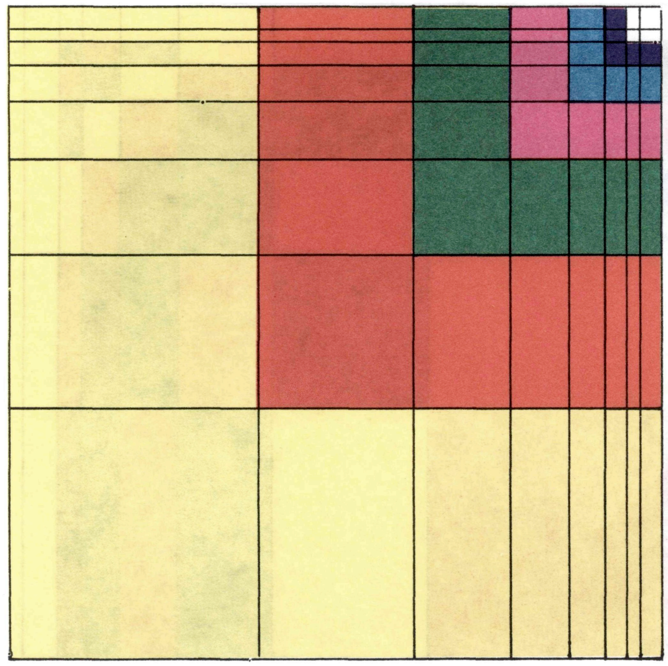

FIGURE 7: $1/\Phi + 1/\Phi^3 + 1/\Phi^5 + \cdots + 1/\Phi^{2n-1} + \cdots = 1$

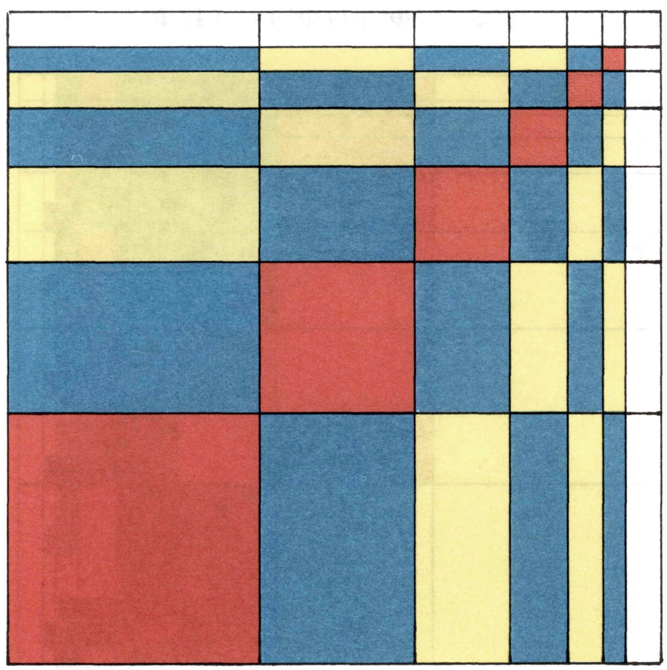

FIGURE 8: $1/\Phi^2 + 1/\Phi^4 + 1/\Phi^6 + \cdots + 1/\Phi^{2n} + \cdots = 1/\Phi$

$1/\Phi^4 + 2/\Phi^6 + 3/\Phi^8 + \cdots + n/\Phi^{2n+2} + \cdots = 1/\Phi^2$

$1/\Phi^3 + 2/\Phi^5 + 3/\Phi^7 + \cdots + n/\Phi^{2n+1} + \cdots = 1/\Phi$

REFERENCES

1. Brother Alfred Brousseau. "Fibonacci Numbers and Geometry." *The Fibonacci Quarterly* **10.3** (1972):303-18.
2. Marjorie Bicknell & Verner E. Hoggatt, Jr. "Golden Triangles, Rectangles, and Cuboids." *The Fibonacci Quarterly* **7.1** (1969):73-91.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*, Chs.3 and 4. Boston: Houghton-Mifflin, 1969; rpt. Santa Clara, Calif.: The Fibonacci Association, 1980.

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