SQUARES OF SECOND-ORDER LINEAR RECURRENCE SEQUENCES

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INTRODUCTION

Let us call a sequence $\{T_n\}$ $(n \geq 0)$ an "m^{th}-order sequence" if $\{T_n\}$ $(n \geq 0)$ satisfies an m^{th}-order linear recurrence relation with constant integer coefficients. (We allow constant terms to appear in our recurrence relations.) From now on we shall generally write simply $\{T_n\}$ rather than $\{T_n\}$ $(n \geq 0)$. It is well known ([2], [3]) that if $\{T_n\}$ is a second-order sequence then the sequence of squares $\{T_n^2\}$ is a third-order sequence. (It is also easy to show this directly.) It would be of interest to be able to describe all second-order sequences $\{T_n\}$ such that $\{T_n^2\}$ is a *second*-order sequence.

In this note we do this for certain *homogeneous* sequences $\{T_n\}$. That is, we assume that $\{T_n\}$ satisfies a recurrence of the form $T_0 = a, T_1 = b, T_{n+1} = cT_n - dT_{n-1}, n \geq 1$, where $a, b, c \neq 0$, $d \neq 0$ are integers, $ab \neq 0$, and $x^2 - cx + d = 0$ has distinct roots. It then turns out that $\{T_n^2\}$ satisfies a second-order linear recurrence (which we describe in Theorem 6) if and only if $d = 1$.

As an illustration of this, consider the sequence 1, 2, 7, 26, 97, 362, ... which satisfies the second-order recurrence $B_0 = 1, B_1 = 2, B_{n+1} = 4B_n - B_{n-1}, n \geq 1$. The sequence of squares $1^2, 2^2, 7^2, 26^2, 97^2, 362^2, \ldots$ satisfies the second-order recurrence $S_0 = 1, S_1 = 4, S_{n+2} = 14S_{n+1} - S_n - 6$.

We also consider second-order sequences $\{T_n\}$ such that a slight perturbation of the sequence of squares $\{T_n^2\}$ is a second-order sequence. For example, the sequence 1, 1, 3, 7, 17, 41, 99, ... satisfies the second-order recurrence $B_0 = B_1 = 1, B_{n+2} = 2B_{n+1} + B_n$, and the "perturbed" sequence of squares $1^2, 1^2+1, 3^2, 7^2+1, 17^2, 41^2+1, 99^2, \ldots$ satisfies the second-order recurrence $S_0 = 1$, $S_1 = 2$, $S_{n+2} = 6S_{n+1} - S_n - 2$.

We begin with some special cases using elementary techniques. Then, in the last section, we handle the general case using an old result of E. S. Selmer [3] which states: if $T_{n+1} = AT_n + BT_{n-1}$, $n \geq 1$, and $x^2 - Ax - B = (x-\alpha)(x-\beta), \alpha \neq \beta$, then $T_{n+1}^2 = CT_n^2 + DT_{n-1}^2 + ET_{n-2}^2, n \geq 2$, where $x^3 - Cx^2 - Dx - E = (x-\alpha^2)(x-\beta^2)(x-\alpha\beta)$.

MAIN RESULTS

We begin with some special cases for which we will use the following Lemma.

Lemma: Let $p \geq 4$ be any integer, let $\delta = \sqrt{\frac{p}{4}} + \sqrt{\frac{p}{4} - 1}$, and let $S_n = (\delta^n + \frac{1}{\delta^n})^2$, $n \geq 0$. Then these numbers S_n satisfy the following identities.

(a) For all $0 \le m \le n$, $(S_n - 2)(S_m - 2) = S_{n+m} + S_{n-m} - 4$.
[In particular, $(S_n - 2)^2 = S_{2n}$, so that S_{2n} is always a perfect square.]

(b) For all $0 \le m \le n$, $m \equiv n \pmod 2$, $S_n S_m = \left(S_{(n+m)/2} + S_{(n-m)/2} - 4\right)^2$.
[In particular, $S_{n+k}S_{n-k} = (S_n + S_k - 4)^2$ and $pS_{2n+1} = S_1 S_{2n+1} = (S_n + S_{n+1} - 4)^2$, so that S_{2n+1} is always a perfect square provided p is a perfect square.]

(c) For all $0 \le m \le n$, $m \equiv n \pmod 2$, $(S_n - 4)(S_m - 4) = \left(S_{(n+m)/2} - S_{(n-m)/2}\right)^2$.
[In particular, $(p-4)(S_{2n+2} - 4) = (S_1 - 4)(S_{2n+1} - 4) = (S_{n+1} - S_n)^2$, so that $S_{2n+1} - 4$ is always a perfect square provided $p - 4$ is a perfect square.]

(d) $S_{n+2} = (p-2)S_{n+1} - S_n - 2(p-4), n \ge 0$.

Proof: We prove part (d) in detail. The proofs of parts (a), (b), and (c) are very similar, and are omitted.

Note that $\frac{1}{\delta} = \sqrt{\frac{p}{4}} - \sqrt{\frac{p}{4} - 1}$, so that $(\delta + \frac{1}{\delta})^2 = p$. Then

$$pS_{n+1} = \left(\delta + \frac{1}{\delta}\right)^2 S_{n+1} = \left[\left(\delta + \frac{1}{d}\right)\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)\right]^2 = \left[\left(\delta^{n+2} + \frac{1}{\delta^{n+2}}\right) + \left(\delta^n + \frac{1}{\delta^n}\right)\right]^2$$

$$= S_{n+2} + S_n + 2\left[\delta^{2n+2} + \frac{1}{\delta^{2n+2}} + \delta^2 + \frac{1}{\delta^2}\right]$$

$$= S_{n+2} + S_n + 2\left[\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)^2 - 2 + \left(\delta + \frac{1}{\delta}\right)^2 - 2\right]$$

$$= S_{n+2} + S_n + 2S_{n+1} + 2(p-4),$$

that is, $S_{n+2} = (p-2)S_{n+1} - S_n - 2(p-4), n \ge 0$.

Theorem 1: Let $d \ge 3$ be an integer. Define the sequence $\{B_n\}$ $(n \ge 0)$ by $B_0 = 2, B_1 = d$, $B_{n+2} = dB_{n+1} - B_n$, $n \ge 0$. Then the sequence of squares $\{B_n^2\}$ $(n \ge 0)$ satisfies the second-order recurrence

$$B_{n+2}^2 = (d^2 - 2)B_{n+1}^2 - B_n^2 - 2(d^2 - 4), \ n \ge 0.$$

Proof: Solving the recurrence $B_0 = 2, B_1 = d, B_{n+2} = dB_{n+1} - B_n$ in the usual way gives

$$B_n = \delta^n + \frac{1}{\delta^n}, n \ge 0, \text{ where } \delta = \sqrt{\frac{d^2}{4}} + \sqrt{\frac{d^2}{4} - 1}, \ \ \frac{1}{\delta} = \sqrt{\frac{d^2}{4}} - \sqrt{\frac{d^2}{4} - 1}.$$

Let us now simplify the notation by setting $S_n = B_n^2, n \ge 0$. Then $S_n = (\delta^n + \frac{1}{\delta^n})^2$, $n \ge 0$, and by part (d) of the Lemma (with $p = d^2$), $S_{n+2} = (d^2 - 2)S_{n+1} - S_n - 2(d^2 - 4)$, $n \ge 0$.

Now we give a second-order sequence whose squares, when slightly perturbed, form a second-order sequence.

Theorem 2: Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}$ $(n \geq 0)$ by $C_0 = 2, C_1 = d,$ $C_{n+2} = dC_{n+1} + C_n,$ $n \geq 0$. Let $S_{2n} = C_{2n}^2, S_{2n+1} = C_{2n+1}^2 + 4,$ $n \geq 0$. Then

$$S_{n+2} = (d^2+2)S_{n+1} - S_n - 2d^2, \; n \geq 0.$$

Proof: Solving the recurrence $C_0 = 2, C_1 = d, C_{n+2} = dC_{n+1} + C_n$ $(n \geq 0)$ in the usual way gives

$$C_n = \delta^n + \left(\frac{-1}{n}\right)^n, \text{ where } \delta = \sqrt{\frac{d^2}{4}+1} + \sqrt{\frac{d^2}{4}}, \; \frac{1}{\delta} = \sqrt{\frac{d^2}{4}+1} - \sqrt{\frac{d^2}{4}}.$$

Then $S_{2n} = C_{2n}^2 = (\delta^{2n} + \frac{1}{\delta^{2n}})^2,$ $S_{2n+1} = C_{2n+1}^2 + 4 = (\delta^{2n+1} + \frac{1}{\delta^{2n+1}})^2,$ $n \geq 0$.

Since $(\delta + \frac{1}{\delta})^2 = d^2 + 4$, we obtain

$$(d^2+4)S_{n+1} = \left[\left(\delta + \frac{1}{\delta}\right)\left(\delta^{n+1} + \frac{1}{\delta^{n+1}}\right)\right]^2,$$

and the calculations used in the proof of part (d) of the Lemma now give

$$S_{n+2} = (d^2+2)S_{n+1} - S_n - 2d^2, \; n \geq 0.$$

Corollary 1: Let $S_{2n} = L_{2n}^2, S_{2n+1} = L_{2n+1}^2 + 4,$ $n \geq 0$, where $\{L_n\}$ is the Lucas sequence. Then $S_{n+2} = 3S_{n+1} - S_n - 2$.

Proof: This is the case $d = 1$ of Theorem 2.

Corollary 2: Let $T_{2n} = F_{2n}^2 + \frac{4}{5}, T_{2n+1} = F_{2n+1}^2,$ $n \geq 0$, where $\{F_n\}$ is the Fibonacci sequence. Then $T_{n+2} = 3T_{n+1} - T_n - 2,$ $n \geq 0$.

Proof: This follows from Corollary 1 and the identity $5F_n^2 = L_n^2 - 4(-1)^n$ (see [1], p. 56).

If we now write $\delta = \sqrt{s} - \sqrt{s-1}, S_n = \frac{1}{4}(\delta^n + \frac{1}{\delta^n})^2,$ $n \geq 0$, we obtain, just as in the Lemma, $S_0 = 1,$ $S_1 = s, S_{n+2} = 4(s-2)S_{n+1} - S_n - 2(s-1),$ $n \geq 0$.

The following two results can now be proved in essentially the same way as Theorems 1 and 2.

Theorem 3: Let $d \geq 2$ be an integer. Define the sequence $\{B_n\}$ $(n \geq 0)$ by $B_0 = 1, B_1 = d,$ $B_{n+2} = 2dB_{n+1} - B_n,$ $n \geq 0$. Then the sequence of squares $\{B_n^2\}$ $(n \geq 0)$ satisfies the second-order recurrence $B_{n+2}^2 = (4d^2-2)B_{n+1}^2 - B_n^2 - 2(d^2-1),$ $n \geq 0$.

Theorem 4: Let $d \geq 1$ be an integer. Define the sequence $\{C_n\}$ $(n \geq 0)$ by $C_0 = 1, C_1 = d,$ $C_{n+2} = 2dC_{n+1} + C_n,$ $n \geq 0$. Assume $S_{2n} = C_{2n}^2, S_{2n+1} = C_{2n+1}^2,$ $n \geq 0$, then $S_{n+2} = (4D^2+2)S_{n+1} - S_n - 2d^2,$ $n \geq 0$.

We now turn to the more general homogeneous case.

Theorem 5: Let $a, b, c \neq 0, d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$. Let $B_0 = a, B_1 = b$, $B_{n+1} = cB_n - dB_{n-1}$, $n \geq 1$. Then $B_{n+1}^2 = (c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + 2(b^2 + a^2 d - abc)d^n$, $n \geq 1$.

Proof: Let α, β be the roots of $x^2 - cx + d = 0$. Then $\alpha, \beta = \frac{1}{2}(c \pm \sqrt{c^2 - 4d})$, $\alpha \neq \pm\beta$, $\alpha^2, \beta^2 = \frac{1}{2}(c^2 - 2d \pm c\sqrt{c^2 - 4d}), \alpha\beta = d$. Also $\alpha^2 \neq \beta^2 \neq d$, since $c \neq 0, d \neq 0, c^2 \neq 4d$.

According to the result of Selmer stated in the Introduction, there are constants A, B, C such that $B_n^2 = A\alpha^{2n} + B\beta^{2n} + Cd^n$, $n \geq 0$.

Solving the system

$$\begin{cases} a^2 = B_0^2 = A + B + C \\ b^2 = B_1^2 = A\alpha^2 + B\beta^2 + Cd \\ (bc - ad)^2 = B_2^2 = A\alpha^4 + B\beta^4 + Cd^2 \end{cases}$$

for C gives

$$C = \frac{2(b^2 + a^2 d - abc)}{4d - c^2}.$$

Using $(c^2 - 2d)\alpha^{2n} - d^2\alpha^{2n-2} = \alpha^{2n+2}$ and $(c^2 - 2d)\beta^{2n} - d^2\beta^{2n-2} = \beta^{2n+2}$ gives

$$(c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + C[(c^2 - 2d)d^n - d^{n+1}] + ed^n.$$

Now choosing e so that $C[(c^2 - 2d)d^n - d^{n+1}] + ed^n = Cd^{n+1}$ [namely, $e = C(4d - c^2) = 2(b^2 + a^2 d - abc)$] finally gives

$$(c^2 - 2d)B_n^2 - d^2 B_{n-1}^2 + ed^n = A\alpha^{2n+2} + B\beta^{2n+2} + Cd^{n+1} = B_{n+1}^2,$$

which completes the proof.

Remark: The result of Theorem 5 appears in [4].

Applying Theorem 5 to the question raised in the Introduction, we immediately get the following result.

Theorem 6: Let $a, b, c \neq 0, d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$. Let $B_0 = a, B_1 = b, B_{n+1} = cB_n - dB_{n-1}$, $n \geq 1$. Then the sequence of squares $\{B_n^2\}$ $(n \geq 0)$ satisfies a second-order linear recurrence (with constant coefficients) if and only if $d = 1$, in which case

$$B_{n+1}^2 = (c^2 - 2)B_n^2 - B_{n-1}^2 + 2(b^2 + a^2 - abc),\ n \geq 1.$$

Our final result is the general version of Theorem 2, in which we consider a perturbation of the sequence of squares.

Theorem 7: Let $a, b, c \neq 0, d \neq 0$ be integers, with $ab \neq 0$ and $c^2 \neq 4d$, such that $e = \frac{4(a^2 + abc - b^2)}{c^2 + 4}$ is an integer. Define the sequence $\{B_n\}$ $(n \geq 0)$ by $B_0 = a, B_1 = b, B_{n+1} = cB_n + B_{n-1}$, $n \geq 1$. Let

$S_{2n} = B_{2n}^2$, $S_{2n+1} = B_{2n+1}^2 + e$, $n \geq 0$. Then $\{S_n\}$ $(n \geq 0)$ satisfies the second-order recurrence

$$S_{n+1} = (c^2+2)S_n - S_{n-1} + 2e + 2(b^2 - a^2 - abc),\ n \geq 1.$$

Proof: This is a direct application of Theorem 5 with $d = -1$, according to which

$$B_{n+1}^2 = (c^2+2)B_n^2 - B_{n-1}^2 + 2(b^2 - a - abc)(-1)^n.$$

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