RECURRENCE SEQUENCES AND BERNOULLI POLYNOMIALS OF HIGHER ORDER

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1. It is well known that a general, linear sequence $S_n(p,q)$ $(n=0,1,2,\dots)$ of order 2 is defined by the recurrence relation

$$S_n(p,q)=pS_{n-1}(p,q)-qS_{n-2}(p,q)$$

with S_0, S_1, p, q arbitrary and $\Delta=p^2-4q>0$ [5].

In particular, if $S_0=0, S_1=1$ or $S_0=2, S_1=p$, we have the generalized Fibonacci (Lucas) sequences $S_n(p,q)=U_n(p,q)$ or $S_n(p,q)=V_n(p,q)$.

If x_1, x_2 $(x_1>x_2)$ designate the roots of $x^2-px+1=0$, it is easy to prove that

$$U_n(p,q)=\frac{x_1^n-x_2^n}{x_1-x_2},\quad V_n(p,q)=x_1^n+x_2^n \tag{1}$$

and, moreover, the general term of the sequence $S_n(p,q)$ may be expressed as a linear combination of the general terms of the Fibonacci and Lucas sequences by the formula

$$S_n(p,q)=\left(S_1-\tfrac{1}{2}pS_0\right)U_n(p,q)+\tfrac{1}{2}S_0V_n(p,q). \tag{2}$$

We assume $S_0=k, S_1=\frac{1}{2}pk+\left(x-\frac{1}{2}k\right)\Delta^{\frac{1}{2}}$, and according to equations (1) and (2), we deduce

$$S_n(x;p,q)=\left(x-\tfrac{1}{2}k\right)\Delta^{\frac{1}{2}}U_n(p,q)+\tfrac{1}{2}kV_n(p,q), \tag{3}$$

$$S_n(x;p,q)=xx_1^n+(k-x)x_2^n. \tag{4}$$

For the sake of brevity, from here on we write U_n, V_n, and $S_n(x)$ for $U_n(p,q), V_n(p,q)$, and $S_n(x;p,q)$.

2. From equation (3) we have

$$S_n(x)+S_n(k-x)=\frac{1}{2^{m-1}}\sum_{r=0}^{[m/2]}\binom{m}{2r}\Delta^r U_n^{2r}V_n^{m-2r}k^{m-2r}(2x-k)^{2r}, \tag{5}$$

and from (4) we get

$$S_n^m(x)+S_n^m(k-x)=\sum_{r=0}^{m}\binom{m}{2r}q^{nr}(x_1^{n(m-2r)}+x_2^{n(m-2r)})x^r(k-x)^{m-r}.$$

Then we have

$$S_n^{2m}(x)+S_n^{2m}(k-x)=\sum_{r=0}^{2m}\binom{2m}{r}q^{nr}(x_1^{2n(m-r)}+x_2^{2n(m-r)})x^r(k-x)^{2m-r}=\sum_{r=0}^{m}+\sum_{r=m+1}^{2m}$$

$$=\sum_{r=0}^{2m}\binom{2m}{r}q^{nr}(x_1^{2n(m-r)}+x_2^{2n(m-r)})x^r(k-x)^{2m-r}+\sum_{s=0}^{m-1}\binom{2m}{s}q^{ns}(x_1^{2n(m-s)}+x_2^{2n(m-s)})x^{2m-s}(k-x)^s$$

$$
\begin{aligned}
&= 2\binom{2m}{m} q^{mn} x^m (k-x)^m + \sum_{r=0}^{m-1} \binom{2m}{r} q^{nr} (x_1^{2n(m-r)} + x_2^{2n(m-r)})(x^r (k-x)^{2m-r} + x^{2m-r}(k-x)^r) \\
&= 2\binom{2m}{m} q^{mn} x^m (k-x)^m + \sum_{r=0}^{m-1} \binom{2m}{r} q^{nr} V_{2n(m-1)} (x^r (k-x)^{2m-r} + x^{2m-r}(k-x)^r).
\end{aligned}
\tag{6}
$$

We can similarly find the analogous formula

$$
S_n^{2m+1}(x) + S_n^{2m+1}(k-x) = \sum_{r=0}^{m} \binom{2m+1}{r} q^{nr} V_{n(2m-2r+1)} (x^r (k-x)^{2m-r+1} + x^{2m-r+1}(k-x)^r). \tag{7}
$$

We also give the difference formulas

$$
S_n^m(x) - S_n^m(k-x) = \frac{\Delta^{\frac{1}{2}}}{2^{m-1}} \sum_{r=0}^{[(m-1)/2]} \binom{m}{2r+1} \Delta^r U_n^{2r+1} V_n^{m-2r-1} k^{m-2r-1} (2x-k)^{2r+1}, \tag{8}
$$

$$
S_n^m(x) + S_n^m(k-x) = \Delta^{\frac{1}{2}} \sum_{r=0}^{[(m-1)/2]} \binom{m}{r} q^{nr} U_{n(m-2r)} [x^{m-r}(k-x)^r - x^r (k-x)^{m-r}]. \tag{9}
$$

We end this section with the generating functions

$$
\sum_{r=0}^{\infty} \frac{t^r}{r!} U_{nr} = \frac{1}{\Delta^{\frac{1}{2}}} (\exp(tx_1^n) - \exp(tx_2^n)), \tag{10}
$$

and

$$
\sum_{r=0}^{\infty} \frac{t^r}{r!} V_{nr} = \exp(tx_1^n) + \exp(tx_2^n). \tag{11}
$$

3. We recall that Bernoulli polynomials of higher order $B_n^{(k)}(x)$ are defined by the generating expansion (see [2])

$$
\sum_{r=0}^{\infty} \frac{t^r}{r!} B_r^{(k)}(x) = \left(\frac{t}{e^t - 1}\right)^k e^{tx}, \ |t| < 2\pi, \tag{12}.
$$

the usual Bernoulli polynomials by $B_n(x) = B^{(1)}(x)$, and the Bernoulli numbers by $B_n = B_n(0)$. We also recall that

$$
B_{2n+1} = 0 \ (n > 0) \ \text{ and } \ B_n^{(k)}(k-x) = (-1)^n B_n^k(x), \tag{13}
$$

are usually called the complementary argument theorem (see [3]).

From (12), replacing t with $\Delta^{\frac{1}{2}} U_n t$, we have

$$
\begin{aligned}
&\sum_{r=0}^{\infty} \frac{\left(\Delta^{\frac{1}{2}} U_n t\right)^r}{r!} B_r^{(k)}(x) = \frac{\left(\Delta^{\frac{1}{2}} U_n t\right)^k}{\left(\exp(t(x_1^n - x_2^n))\right)^k} \exp(tx(x_1^n - x_2^n)) \\
&= \frac{(\Delta^{\frac{1}{2}} U_n t)^k}{\left(\exp(tx_1^n) - \exp(x_2^n)\right)^k} \exp(t(xx_1^n + (k-x)x_2^n)) = \frac{(\Delta^{\frac{1}{2}} U_n t)^k}{(\exp(tx_1^n) - \exp(x_2^n))^k} \exp(tS_n(x)).
\end{aligned}
$$

Therefore,

$$(\exp(tx_1^n)-\exp(tx_2^n))^k\sum_{r=0}^{\infty}\frac{(\Delta^{\frac{1}{2}}U_nt)^r}{r!}B_r^{(k)}(x)=(\Delta^{\frac{1}{2}}U_nt)^k\exp(tS_n(x))$$

and using (10) it follows that

$$\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}U_{nr}\right)^k\sum_{r=0}^{\infty}\frac{(\Delta^{\frac{1}{2}}U_nt)^r}{r!}B_r^{(k)}(x)=U_n^kt^k\exp(tS_n(x)),$$

and we have

$$\left[\sum_{r=0}^{\infty}\left(\sum_{r_1+r_2+\cdots+r_k=r}\frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!}\right)t^r\right]\sum_{r=0}^{\infty}\frac{(\Delta^{\frac{1}{2}}U_nt)^r}{r!}B_r^{(k)}(x)=U_r^{(k)}t^k\exp(tS_n(x)).$$

Expanding the product in the left-hand side and comparing powers of t, we find

$$\sum_{r=0}^{m-1}\binom{m}{r}\Delta^{\frac{r}{2}}U_n^rB_r^{(k)}(x)(m-r)!\sum_{r_1+r_2+\cdots+r_k=m-r}\frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!}=(m)_kU_n^kS_n^{m-k}(x),\quad(m\geq k).\tag{14}$$

Now replace x with $k-x$ in equation (14) and make use of (13) to obtain

$$\sum_{r=0}^{m-1}(-1)^r\binom{m}{r}\Delta^{\frac{r}{2}}U_n^rB_r^{(k)}(x)(m-r)!\sum_{r_1+r_2+\cdots+r_k=m-r}\frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!}=(m)_kU_n^kS_n^{m-k}(k-x),\quad(m\geq k).\tag{15}$$

Adding equations (14) and (15) and using (5), (6), and (7) we find

$$\begin{aligned}&\sum_{r=0}^{[(m-1)/2]}\binom{m}{2r}\Delta^rU_n^{2r}B_{2r}^k(x)(m-2r)!\sum_{r_1+r_2+\cdots+r_k=m-2r}\frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!}\\&=\frac{1}{2^{m-k}}(m)_kU_n^k\sum_{r=0}^{[(m-k)/2]}\binom{m-k}{2r}\Delta^rU_n^{2r}k^{m-2r-k}U_n^{m-2r-k}(2x-k)^{2r}\end{aligned}\tag{16}$$

$$\begin{aligned}&=\frac{1+(-1)^{m-k}}{2}(m)_kU_n^k\binom{m-k}{[(m-k)/2]}q^{[(m-k)/2]}(x(k-x))^{[(m-k)/2]}+\\&+\frac{1}{2}(m)_kU_n^k\sum_{r=0}^{[(m-k-1)/2]}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}[x(k-x)^{m-k-r}+x^{m-k-r}(k-x)^r].\end{aligned}\tag{17}$$

Subtracting equations (14) and (15) and using (8) and (9) gives

$$\begin{aligned}&\sum_{r=0}^{[(m-2)/2]}\binom{m}{2r+1}\Delta^rU_n^{2r+1}B_{2r+1}^{(k)}(x)(m-2r-1)!\sum_{r_1+r_2+\cdots+r_k=m-2r-1}\frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!}\\&=\frac{1}{2^{m-k}}(m)_kU_n^k\sum_{r=0}^{[(m-k-1)/2]}\binom{m-k}{r}q^{nr}U_{n(m-2r-k)}[x^{m-r-k}(k-x)^r-x^r(k-x)^{m-r-k}].\end{aligned}\tag{18}$$

4. If we take $x = \frac{k}{2}$ in equation (16), we get

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)}\left(\frac{k}{2}\right)(m-2r)! \sum_{r_1+r_2+\cdots r_k=m-2r} \frac{U_{nr_1}}{r_1!} \cdots \frac{U_{nr_k}}{r_k!} = \frac{1}{2^{m-k}}(m)_k U_n^k k^{m-k} V_n^{m-k}. \quad (19)$$

Taking $k = 1$ in equation (19) and recalling that $B_{2n}\left(\frac{1}{2}\right) = \left(\frac{1}{2^{2n-1}} - 1\right)B_{2n}$ (see [5]), we have

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U^{2r}\left(\frac{1}{2^{2r-1}} - 1\right) B_{2r} U_{n(m-2r)} = \frac{m}{2^{m-1}} U_n V_n^{m-1}. \quad (20)$$

If we make $x = 0$ in equation (17), we get

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)} (m-2r)! \sum_{r_1+r_2+\cdots+r_k=m-2r} \frac{U_{nr_1}}{r_1!} \cdots \frac{U_{nr_k}}{r_k!} = \frac{1}{2}(m)_k U_n^k V_{(m-k)}. \quad (21)$$

From expression (21), recalling that $B_k^{(n+1)} = \left(1 - \frac{k}{n}\right)B_k^{(n)} - kB_{k-1}^{(n)}$ (see [4]) and taking $k = 1, 2, 3$, we get, successively,

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r} U_{n(m-2r)} = \frac{1}{2} m U_n V_{n(m-1)}, \quad (22)$$

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} (B_{2r} - 2rB_{2r} - 2rB_{2r-1}) \sum_{i=0}^{m-2r} \binom{m-2r}{i} U_{ni} U_{n(m-2r-1)} = \frac{1}{2} m(m-1) U_n^2 V_{n(m-2)}, \quad (23)$$

$$\sum_{r=0}^{[(m-1)/2]} \binom{m}{2r} \Delta^r U_n^{2r} ((2r-1)(r-1)B_{2r} + r(4r-5)B_{2r-1} + 2r(2r-1)B_{2r-2}) \sum_{i+j+k=m-2r} \binom{m-2r}{i,\, j,\, k} U_{ni} U_{nj} U_{nk}$$
$$= \frac{1}{2} m(m-1)(m-2) U_n^3 V_{n(m-3)}, \quad (24)$$

where $\binom{m-2r}{i,\, j,\, k}$ is the multinomial coefficient (see [1]).

With $p = 1$, $q = -1$, we get the Fibonacci and Lucas sequences $U_0 = 0, U_1 = 1, \ldots, U_n(1, -1) = F_n, \ldots$ and $V_0 = 2, V_1 = 1, \ldots, V_n(1, -1) = L_n, \ldots$, and from equation (19) we get Kelisky's formula (see [2])

$$\sum_{r=0}^{[m/2]} 5^r \binom{m}{2r} B_{2r} F_n^{2r} F_{n(m-2r)} = \frac{m}{2} F_n L_{n(m-1)}. \quad (25)$$

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