ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-503 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let $\mathcal{S}$ be the set of functions $F: C^3 \to C$ (C is the complex plane) satisfying the following formal properties:

$$xyz\,F(x, x^3y, x^3y^2z) = F(x, y, z); \tag{1}$$

$$F(x^{-1}, y, z^{-1}) = F(x, y, z). \tag{2}$$

Formally define the functions U and V as follows:

$$U(x, y, z) = \sum x^{n^3} y^{n^2} z^n \quad \text{(summed over all integers } n\text{);} \tag{3}$$

$$V(x, y, z) = \prod_{n=1}^{\infty} (1 - y^{2n} A(x))(1 + x^{3n^2-3n+1} y^{2n-1} z)(1 + x^{-3n^2+3n-1} y^{2n-1} z^{-1}), \tag{4}$$

where

$$A(x) = \frac{\sum x^{3m} o_m}{\sum x^{3m} e_m} \quad \text{(summed over all integers } m\text{),} \tag{5}$$

$$o_m = \tfrac{1}{2}(1 - (-1)^m), \quad e_m = \tfrac{1}{2}(1 + (-1)^m). \tag{6}$$

Show that, at least formally,

$$U \in \mathcal{S}, \quad V \in \mathcal{S}; \tag{7}$$

$$A(1) = 1; \tag{8}$$

$$U(1, y, z) = V(1, y, z). \tag{9}$$

Prove or disprove that $U(x, y, z) \equiv V(x, y, z)$ identically. Can $U(x, y, z)$ be factored into an infinite product?

H-504 ***Proposed by Z. W. Trzaska, Warsaw, Poland***

Given a sequence of polynomials in complex variable $z \in C$ defined recursively by

(i) $\qquad R_{k+1}(z) = (3 + z)R_k(z) - R_{k-1}(z), \quad k = 0, 1, 2, \ldots,$

with $R_0(z) = 1$ and $R_1(z) = (1 + z)R_0$.

Prove that

(ii) $$R_k(0) = F_{2k+1},$$

where F_ℓ, $\ell = 0, 1, 2, \ldots$, denotes the ℓ^{th} term of the Fibonacci sequence.

H-505 ***Proposed by Juan Pla, Paris, France***

Edouard Lucas once noted: "On ne connaît pas de formule simple pour la somme des cubes du binôme" [No simple formula is known for the sum of the cubes of the binomial coefficients] (see Edouard Lucas, *Théorie des Nombres*, Paris, 1891, p. 133, as reprinted by Jacques Gabay, Paris, 1991).

The following problem is designed to find *closed*, if not quite "simple," formulas for the sum of the cubes of all the coefficients of the binomial $(1+x)^n$.

1) Prove that

$$\sum_{p=0}^{p=n} \binom{n}{p}^3 = \frac{2^n}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \{1 + \cos\varphi + \cos\theta + \cos(\varphi+\theta)\}^n d\theta\, d\varphi.$$

2) Prove that

$$\sum_{p=0}^{p=n} \binom{n}{p}^3 = \frac{8^n}{\pi^2} \int_0^{\pi} \int_0^{\pi} \{\cos\varphi \cos\theta \cos(\varphi+\theta)\}^n d\theta\, d\varphi.$$

SOLUTIONS

Sum Problem

H-489 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 32, no. 4, August 1994)

Define the sequences of Pell numbers and Pell-Lucas numbers by

$$P_0 = 0,\ P_1 = 1,\ P_{k+2} = 2P_{k+1} + P_k,$$
$$Q_0 = 2,\ Q_1 = 2,\ Q_{k+2} = 2Q_{k+1} + Q_k.$$

Show that

(a) $$\sum_{n=1}^{\infty} \frac{F_{2^n} Q_{2^n}}{8(L_{2^n} P_{2^n})^2 - 5(F_{2^n} Q_{2^n})^2} = \frac{1}{12},$$

(b) $$\sum_{n=1}^{\infty} \frac{L_{2^n} P_{2^n}}{8(L_{2^n} P_{2^n})^2 - 5(F_{2^n} Q_{2^n})^2} = \frac{8 - 3\sqrt{2}}{48}.$$

Solution by Norbert Jensen, Kiel, Germany

Step (1): Solving the characteristic equation and determining the explicit formulas for $(P_k)_{k\in\mathbb{N}_0}$ and $(Q_k)_{k\in\mathbb{N}_0}$, we obtain

$$P_k = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^k - (1-\sqrt{2})^k), \tag{1.1}$$

$$Q_k = (1+\sqrt{2})^k + (1-\sqrt{2})^k \quad \text{for all } k \in \mathbb{N}_0. \tag{1.2}$$

Let $\gamma := 1+\sqrt{2}$, $\delta := 1-\sqrt{2}$. As γ and δ are the zeros of the characteristic polynomial, it follows that

$$\gamma\delta = -1 \text{ and } \gamma + \delta = 2. \tag{1.3}$$

Since the sequences $(\gamma^n)_{n\in\mathbb{N}_0}$ and $(\delta^n)_{n\in\mathbb{N}_0}$ satisfy the same recursion as $(P_k)_{k\in\mathbb{N}_0}$, it follows by induction that

$$\gamma^n = P_n\gamma + P_{n-1} = P_n\sqrt{2} + (P_n + P_{n-1}) \tag{1.4}$$

$$[\text{and } \delta^n = P_n\delta + P_{n-1} = -P_n\sqrt{2} + (P_n + P_{n-1})] \quad \text{for all } n \in \mathbb{N}.$$

Calculation shows that

$$|\alpha/\gamma| = \alpha/\gamma < 1. \quad [\text{Of course } |\beta/\gamma| < 1.] \tag{1.5}$$

Step (2): We need the following general identity: For all $z \in \mathbb{R}$ such that $|z| < 1$, we have

$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{1 - z^{2^{n+1}}} = \frac{z^2}{1-z^2}.$$

Proof: The series $\sum_{k=1}^{\infty} z^{2k}$ is absolutely convergent with limit $\frac{z^2}{1-z^2}$, provided $|z| < 1$. Hence, we can sum up the terms z^{2k} in arbitrary order without changing the limit or convergence. Now

$$\sum_{k=1}^{\infty} z^{2k} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} z^{2^n(2m+1)} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} z^{2^{n+1}m+2^n},$$

since, on the right side, each term z^{2k} appears exactly once for each positive integer $k \in \mathbb{N}$. Now, for each fixed $n \in \mathbb{N}$, we have

$$\sum_{m=0}^{\infty} z^{2^{n+1}m+2^n} = \frac{z^{2^n}}{1 - z^{2^{n+1}}}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{z^{2^n}}{1-z^{2^{n+1}}}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{1 - z^{2^{n+1}}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} z^{2^n(2m+1)} = \sum_{k=1}^{\infty} z^{2k} = \frac{z^2}{1-z^2}.$$

Q.E.D. Step (2).

Step (3): it is convenient to prove the following two identities:

$$\sqrt{8} \cdot L_{2^n} P_{2^n} - \sqrt{5} \cdot F_{2^n} Q_{2^n} = 2 \cdot \left(\frac{\gamma}{\alpha}\right)^{2^n} \left[1 - \left(\frac{\alpha}{\gamma}\right)^{2^{n+1}}\right], \tag{3.1}$$

$$\sqrt{8}\cdot L_{2^n}P_{2^n}+\sqrt{5}\cdot F_{2^n}Q_{2^n}=2\cdot\left(\frac{\gamma}{\beta}\right)^{2^n}\left[1-\left(\frac{\beta}{\gamma}\right)^{2^{n+1}}\right]. \tag{3.2}$$

Proof: We have

$$\sqrt{8}\cdot L_{2^n}P_{2^n}\underset{(1.1)}{=}(\alpha^{2^n}+\beta^{2^n})(\gamma^{2^n}-\delta^{2^n})=(\alpha\gamma)^{2^n}-(\alpha\delta)^{2^n}+(\beta\gamma)^{2^n}-(\beta\delta)^{2^n}$$

and

$$\sqrt{5}\cdot F_{2^n}Q_{2^n}\underset{(1.2)}{=}(\alpha^{2^n}-\beta^{2^n})(\gamma^{2^n}+\delta^{2^n})=(\alpha\gamma)^{2^n}+(\alpha\delta)^{2^n}-(\beta\gamma)^{2^n}-(\beta\delta)^{2^n}.$$

Hence, using $\alpha\beta=\gamma\delta=-1$, we have

$$\sqrt{8}\cdot L_{2^n}P_{2^n}-\sqrt{5}\cdot F_{2^n}Q_{2^n}=2\cdot((\beta\gamma)^{2^n}-(\alpha\delta)^{2^n})=2\left(\frac{\gamma}{\alpha}\right)^{2^n}\left[1-\left(\frac{\alpha}{\gamma}\right)^{2^{n+1}}\right].$$

This proves (3.1). Similarly, (3.2) is seen to be

$$\sqrt{8}\cdot L_{2^n}P_{2^n}+\sqrt{5}\cdot F_{2^n}Q_{2^n}=2\cdot((\alpha\gamma)^{2^n}-(\beta\delta)^{2^n})=2\left(\frac{\gamma}{\beta}\right)^{2^n}\left[1-\left(\frac{\beta}{\gamma}\right)^{2^{n+1}}\right].$$

Q.E.D. Step (3).

Step (4): Replacing z in Step (2) by α/γ and β/γ, respectively, we obtain the following identities, where all limits exist:

$$A:=\sum_{n=1}^{\infty}\frac{1}{\sqrt{8}\cdot L_{2^n}P_{2^n}-\sqrt{5}\cdot F_{2^n}Q_{2^n}}\underset{(3.1)}{=}\sum_{n=1}^{\infty}\frac{1}{2}\frac{\left(\frac{\alpha}{\gamma}\right)^{2^n}}{1-\left(\frac{\alpha}{\gamma}\right)^{2^{n+1}}}\underset{\substack{(1.5)\\ \text{Step (2)}}}{=}\frac{1}{2}\frac{\left(\frac{\alpha}{\gamma}\right)^{2}}{1-\left(\frac{\alpha}{\gamma}\right)^{2}}=\frac{1}{2}\frac{\alpha^2}{\gamma^2-\alpha^2}.$$

$$B:=\sum_{n=1}^{\infty}\frac{1}{\sqrt{8}\cdot L_{2^n}P_{2^n}+\sqrt{5}\cdot F_{2^n}Q_{2^n}}\underset{(3.2)}{=}\sum_{n=1}^{\infty}\frac{1}{2}\frac{\left(\frac{\beta}{\gamma}\right)^{2^n}}{1-\left(\frac{\beta}{\gamma}\right)^{2^{n+1}}}\underset{\substack{(1.5)\\ \text{Step (2)}}}{=}\frac{1}{2}\frac{\left(\frac{\beta}{\gamma}\right)^{2}}{1-\left(\frac{\beta}{\gamma}\right)^{2}}=\frac{1}{2}\frac{\beta^2}{\gamma^2-\beta^2}.$$

Step (5): Now, using the fact that linear combinations of converging sequences are convergent against the linear combinations of their limits, we obtain:

(a) $$\sum_{n=1}^{\infty}\frac{F_{2^n}Q_{2^n}}{8(L_{2^n}P_{2^n})^2-5(F_{2^n}Q_{2^n})^2}=\frac{1}{2\sqrt{5}}(A-B)\underset{\text{Step (4)}}{=}\frac{1}{4\sqrt{5}}\left[\frac{\alpha^2}{\gamma^2-\alpha^2}-\frac{\beta^2}{\gamma^2-\beta^2}\right]$$

$$=\frac{1}{4\sqrt{5}}\cdot\frac{(\alpha^2-\beta^2)(2\sqrt{2}+3)}{6\sqrt{2}+9}=\frac{F_2}{4\cdot 3}=\frac{1}{12};$$

(b) $$\sum_{n=1}^{\infty}\frac{L_{2^n}P_{2^n}}{8(L_{2^n}P_{2^n})^2-5(F_{2^n}Q_{2^n})^2}=\frac{1}{2\sqrt{8}}(A+B)\underset{\text{Step (4)}}{=}\frac{1}{4\sqrt{8}}\left[\frac{\alpha^2}{\gamma^2-\alpha^2}+\frac{\beta^2}{\gamma^2-\beta^2}\right]$$

$$=\frac{1}{8\sqrt{2}}\cdot\frac{(\alpha^2+\beta^2)\gamma^2-2(\alpha^2\beta^2)}{\gamma^4-(\alpha^2+\beta^2)\gamma^2+\alpha^2\beta^2}=\frac{1}{8\sqrt{2}}\frac{3\gamma^2-2}{\gamma^4-3\gamma^2+1}$$

$$\underset{(1.4)}{=}\frac{1}{8\sqrt{2}}\frac{3(P_2\sqrt{2}+(P_2+P_1))-2}{P_4\sqrt{2}+P_4+P_3-3(P_2\sqrt{2}+P_2+P_1)+1}=\frac{1}{8\sqrt{2}}\frac{6\sqrt{2}+7}{6\sqrt{2}+9}$$

$$=\frac{6\sqrt{2}+7}{8\cdot 9\sqrt{2}+8\cdot 2\cdot 6}=\frac{(6\sqrt{2}+7)(-8\cdot 9\sqrt{2}+8\cdot 2\cdot 6)}{(8\cdot 2\cdot 6)^2-2(8\cdot 9)^2}$$

$$=\frac{(6\sqrt{2}+7)(-3\cdot\sqrt{2}+4)}{8\cdot 3(4^2-2\cdot 3^2)}=\frac{-8+3\sqrt{2}}{-48}=\frac{8-3\sqrt{2}}{48}.$$

Q.E.D.

Also solved by P. Bruckman, B. Popov, and the proposer.

Just So Many

H-490 ***Proposed by A. Stuparu, Vâlcea, Romania***
(Vol. 32, no. 5, November 1994)

Prove that the equation $S(x)=p$, where p is a given prime number, has just 2^{p-2} solutions, all of them in between p and $p!$. [$S(n)$ is the Smarandache Function: the smallest integer such that $S(n)!$ is divisible by n.]

Solution by Paul S. Bruckman, Edmonds, WA

The stated conclusion is incorrect. The correct number of solutions of the equation $S(x)=p$ is $\tau((p-1)!)$, not 2^{p-2}; here, $\tau(n)$ is the counting function of the divisors of n.

If $S(x)=p$, then $x|p!$, but $x\nmid m!$ for all $m<p$. If q is any prime factor of x, then $q\le p$; for, if $q>p$, then $q|p!$, an obvious contradiction. On the other hand, if *all* prime factors of x are less than p, then $x|(p-1)!$, another contradiction. Therefore, $p|x$. Since $p^1\,\|\,p!$, it follows that $x=pr$, where $r|(p-1)!$.

Conversely, if $x=pr$, where $r|(p-1)!$, then $S(x)\ge p$, since $x\nmid m!$ for all $m<p$. Also, $S(x)\le p$, since $x|p!$. Consequently, $S(x)=p$.

We have proven the following proposition:

$$S(x)=p \text{ iff } x=pr \text{ where } r|(p-1)!. \qquad (*)$$

It follows that the number of solutions of the equation $S(x)=p$ is precisely $\tau((p-1)!)$. A brief table of the first few values of $\tau((p-1)!)$ is on the following page along with values of 2^{p-2}, for comparison.

p	$(p-1)$	$\tau((p-1)!)$	2^{p-2}
2	1	1	1
3	2^1	2	2
5	$2^3\cdot 3^1$	8	8
7	$2^4\cdot 3^2\cdot 5^1$	30	32
11	$2^8\cdot 3^4\cdot 5^2\cdot 7^1$	270	512
13	$2^{10}\cdot 3^5\cdot 5^2\cdot 7^1\cdot 11^1$	792	2048
17	$2^{15}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11^1\cdot 13^1$	5376	32768
19	$2^{16}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1$	14688	131072

The proposer may have been misled by the coincidence that $\tau((p-1)!) = 2^{p-2}$ for $p = 2, 3, 5$. However, since $(p-1)! \sim (2\pi / p)^{1/2}(p/e)^p$ (using Stirling's formula), and since the average order of $\tau(n)$ is $\log n$ (a well-known result of number theory), it follows that the average order of $\tau((p-1)!)$ is asymptotic to $p \log p$ (as $p \to \infty$), which is much smaller than 2^{p-2}. It should be mentioned that such average is taken over *all* $n \le p$, not merely primes. However, $n \log n$ is certainly $o(2^n)$ as $n \to \infty$.

Also solved by M. Ballieu, A. Dujella, N. Jensen, H.-J. Seiffert, and the proposer.

Late Acknowledgment: Paul S. Bruckman solved H-459.

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