EXTENSIONS OF THE HERMITE G.C.D. THEOREMS FOR BINOMIAL COEFFICIENTS

H. W. Gould

Department of Mathematics, West Virginia University, PO Box 6310,
Morgantown, WV 26506-6310; Email: gould@math.wvu.edu

Paula Schlesinger

124 Lawson Drive, Spruce Pine, NC 28777

(*Submitted November 1993*)

1. INTRODUCTION

Dickson [1], in his *History of the Theory of Numbers*, attributes the divisibility theorems

$$\frac{n}{(n,k)}\left|\binom{n}{k}\right. \tag{1.1}$$

and

$$\frac{n-k+1}{(n+1,k)}\left|\binom{n}{k}\right., \tag{1.2}$$

where (n,k) denotes the greatest common divisor of n and k to Hermite [9], [10] whose proofs use the Euclidean algorithm. In [5] and [6] the proofs were extended by one of us to generalized binomial coefficients defined by

$$\left\{ n \atop k \right\} = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1} \quad \text{with} \left\{ n \atop 0 \right\} = 1, \tag{1.3}$$

where $\{A_n\}$ is a sequence of integers such that $A_0 = 0$, $A_n \neq 0$ for $n \geq 1$, and such that the ratio in (1.3) is always an integer. Of course, the ordinary binomial coefficients occur when $A_n = n$. When $A_n = F_n$, the n^{th} Fibonacci number, we obtain the well-known "Fibonomial" coefficients. Another very well-known case is when $A_n = q^n - 1$, in which case the coefficients determined by (1.3) are the Gaussian or q-binomial coefficients. The generalized forms of Hermite's theorems obtained in [6] are as follows:

$$\frac{A_n}{(A_n, A_k)}\left|\left\{ n \atop k \right\}\right. \tag{1.4}$$

and

$$\frac{A_{n-k+1}}{(A_{n+1}, A_k)}\left|\left\{ n \atop k \right\}\right. \quad \text{provided } (A_{n+1}, A_k) \mid A_{n-k+1}. \tag{1.5}$$

In this paper we will replace (1.1) and (1.2) by the following theorems:

$$\frac{n}{(n,k)} \text{g.c.d.}\left(\binom{n}{k}, \binom{n-1}{k-1}\right) = \binom{n}{k} \tag{1.6}$$

and

$$\frac{n+1-k}{(n+1,k)} \text{g.c.d.}\left(\binom{n}{k}, \binom{n}{k-1}\right) = \binom{n}{k}, \tag{1.7}$$

so that the explicit quotients in the original Hermite statements are made evident as just g.c.d.'s of two binomial coefficients.

What is more, the corresponding extensions to generalized binomials take the forms

$$\frac{A_n}{(A_n, A_k)} \text{g.c.d.}\left(\left\{ n \atop k \right\}, \left\{ n-1 \atop k-1 \right\}\right) = \left\{ n \atop k \right\} \tag{1.8}$$

and

$$\frac{A_{n-k+1}}{(A_{n+1-k}, A_k)} \text{g.c.d.}\left(\left\{ n \atop k \right\}, \left\{ n \atop k-1 \right\}\right) = \left\{ n \atop k \right\}. \tag{1.9}$$

Note especially now that in (1.9) the greatest common divisor automatically divides A_{n+1-k} whereas in (1.5) this was not necessarily the case.

Some variations of these theorems will be presented so that the problem raised by Gould in [3] (no solution having appeared in the interim in the *Monthly*) will have a better formulation due to the nature of our present attack. That problem asked for a way to unify Hermite's two theorems into a general result of the form

$$\frac{an+bk+c}{(rn+s, uk+v)} \bigg| \binom{n}{k} \tag{1.10}$$

for suitable parameters

The explicit forms (1.6) and (1.7) were obtained by Schlesinger in November 1986, and are now being published here for the first time.

It should be remarked that Dickson [1] also traces (1.1) in a form valid for multinomial coefficients back to Schönemann [15] who, in 1839, used symmetric functions and p^{th} roots of unity.

2. PROOF OF (1.6) AND (1.7) AND VARIATIONS

We need only the simplest properties of the binomial coefficients and the greatest common divisor to see that

$$\begin{aligned} n\,\text{g.c.d.}\left(\binom{n}{k}, \binom{n-1}{k-1}\right) &= \text{g.c.d.}\left(n\binom{n}{k}, n\binom{n-1}{k-1}\right) \\ &= \text{g.c.d.}\left(n\binom{n}{k}, k\binom{n}{k}\right) = \binom{n}{k}\text{g.c.d.}(n, k) \end{aligned}$$

which proves (1.6). Similarly,

$$\begin{aligned} (n+1-k)\,\text{g.c.d.}\left(\binom{n}{k}, \binom{n}{k-1}\right) &= \text{g.c.d.}\left((n+1-k)\binom{n}{k}, (n+1-k)\binom{n}{k-1}\right) \\ &= \text{g.c.d.}\left((n+1-k)\binom{n}{k}, k\binom{n}{k}\right) = \binom{n}{k}\text{g.c.d.}(n+1-k, k) \end{aligned}$$

which proves (1.7), since $\text{g.c.d.}(n+1-k, k) = \text{g.c.d.}(n+1, k)$.

The improvement offered by this approach is that we avoid the use of the Euclidean algorithm which only told us that $\text{g.c.d.}(n, k) = nx + ky$ for *some* integers x and y. What we have now are explicit values and the only property of the g.c.d. used is linearity, i.e., that $n\,\text{g.c.d.}(A, B) = \text{g.c.d.}(nA, nB)$.

In the same manner in which we proved (1.6) and (1.7), the reader may establish the following variations:

$$\frac{n}{(n,k)}\,\text{g.c.d.}\left(\binom{n}{k},\binom{n-1}{k}\right)=\binom{n}{k}; \tag{2.1}$$

$$\frac{n+1-k}{(n+1-k,n+1)}\,\text{g.c.d.}\left(\binom{n}{k},\binom{n+1}{k}\right)=\binom{n}{k}; \tag{2.2}$$

$$\frac{k+1}{(k+1,n-k)}\,\text{g.c.d.}\left(\binom{n}{k},\binom{n}{k+1}\right)=\binom{n}{k}; \tag{2.3}$$

$$\frac{k+1}{(k+1,n+1)}\,\text{g.c.d.}\left(\binom{n}{k},\binom{n+1}{k+1}\right)=\binom{n}{k}. \tag{2.4}$$

Note that in (1.6), (1.7), (2.1), (2.2), (2.3), and (2.4) we have in each case the g.c.d. of $\binom{n}{k}$ and $\binom{n+a}{k+b}$, where a and b assume only the values $-1, 0, +1$ and in some special manner.

Here is a different-looking result, easily verified:

$$(n+1-k)(k+1)\,\text{g.c.d.}\left(\binom{n}{k-1},\binom{n}{k+1}\right)=\binom{n}{k}\text{g.c.d.}(k(k+1),(n-k)(n+1-k)), \tag{2.5}$$

so that if we try to shift a unit in both coefficients we get quadratic factors appearing.

Relation (1.6) may be extended easily by shifting both k and n in the one binomial coefficient by the same amount. Thus, if we let $0\le i\le k$, we find

$$\begin{aligned}
&(n-i)\,\text{g.c.d.}\left(\binom{n}{k}, n(n-1)\cdots(n-i+1)\binom{n-i-1}{k-i-1}\right)\\
&=\text{g.c.d.}\left((n-i)\binom{n}{k}, n(n-1)\cdots(n-i+1)(n-i)\binom{n-i-1}{k-i-1}\right)\\
&=\text{g.c.d.}\left((n-i)\binom{n}{k}, k(k-1)\cdots(k-i)\binom{n-i-1}{k-i-1}\right)\\
&=\binom{n}{k}\text{g.c.d.}(n-i, k(k-1)\cdots(k-i)),
\end{aligned}$$

so that we have proved

$$\frac{n-i}{((n-i),k(k-1)\cdots(k-i))}\,\bigg|\,\binom{n}{k} \tag{2.6}$$

for every i with $0\le i\le k$, and the quotient is

$$\text{g.c.d.}\left(\binom{n}{k}, n(n-1)\cdots(n-i+1)\binom{n-i-1}{k-i-1}\right).$$

The corresponding proof by the original method of Hermite, using the Euclidean algorithm runs as follows. Let $d=\text{g.c.d.}(n-i, k(k-1)\cdots(k-1))$. Then there exist integers x and y such that $(n-i)x+k(k-1)\cdots(k-i)y=d$. Thus,

$$\begin{aligned}((n-i)x+k(k-1)\cdots(k-i)y)\binom{n}{k} &= (n-i)x\binom{n}{k}+k(k-1)\cdots(k-i)y\binom{n}{k}\\ &= (n-i)x\binom{n}{k}+n(n-1)\cdots(n-i)y\binom{n-i-1}{k-i-1},\end{aligned}$$

whence the result follows; however, it does not yield the explicit quotient value.

Two formulas that are similar to (2.5) and which may be verified by the reader are:

$$n(k+1)\,\text{g.c.d.}\left(\binom{n-1}{k},\binom{n+1}{k+1}\right)=\binom{n}{k}\text{g.c.d.}((k+1)(n-k),n(n+1)) \tag{2.7}$$

and

$$n(n+1-k)\,\text{g.c.d.}\left(\binom{n-1}{k-1},\binom{n+1}{k}\right)=\binom{n}{k}\text{g.c.d.}((n+1-k)k,n(n+1)). \tag{2.8}$$

3. EXTENSION TO GENERALIZED BINOMIAL COEFFICIENTS

The proof of (1.8) runs as follows:

$$\begin{aligned}A_n\,\text{g.c.d.}\left(\left\{{n \atop k}\right\},\left\{{n-1 \atop k-1}\right\}\right) &= \text{g.c.d.}\left(A_n\left\{{n \atop k}\right\},A_n\left\{{n-1 \atop k-1}\right\}\right)\\ &= \text{g.c.d.}\left(A_n\left\{{n \atop k}\right\},A_k\left\{{n \atop k}\right\}\right)=\left\{{n \atop k}\right\}\text{g.c.d.}(A_n,A_k),\end{aligned}$$

while for (1.9), we have

$$\begin{aligned}A_{n+1-k}\,\text{g.c.d.}\left(\left\{{n \atop k}\right\},\left\{{n \atop k-1}\right\}\right) &= \text{g.c.d.}\left(A_{n+1-k}\left\{{n \atop k}\right\},A_{n+1-k}\left\{{n \atop k-1}\right\}\right)\\ &= \text{g.c.d.}\left(A_{n+1-k}\left\{{n \atop k}\right\},A_n\left\{{n \atop k}\right\}\right)=\left\{{n \atop k}\right\}\text{g.c.d.}(A_{n+1-k},A_k).\end{aligned}$$

A simple but important application of (1.9) is to show that the Fibonomial Catalan numbers are in fact integers. Let $A_n=F_n=n^{\text{th}}$ Fibonacci number. Then by (1.9), with the substitutions $n\leftarrow 2n, k\leftarrow n$, we have that $\{{n \atop k}\}_F$ is divisible by $F_{n+1}/(F_{n+1},F_n)$. But $(F_{n+1},F_n)=1$, whence the n^{th} Fibonomial Catalan number

$$\frac{1}{F_{n+1}}\left\{{2n \atop n}\right\}_F \tag{3.1}$$

is an integer. This makes a shorter proof than what was done in [4, p. 363].

4. THEOREMS ABOUT LEAST COMMON MULTIPLES

Since $(a,b)[a,b]=ab$, where $[a,b]$ denotes the least common multiple of a and b, for positive integers a and b, we may convert our theorems to statements about least common multiples. Relation (1.7) may be restated as

$$\text{g.c.d.}\left(\binom{n}{k},\binom{n}{k-1}\right)=\frac{(n+1,k)}{n+1-k}\binom{n}{k} \tag{4.1}$$

so that, in terms of least common multiples

$$\text{l.c.m.}\left(\binom{n}{k}, \binom{n}{k-1}\right) = \frac{n+1-k}{(n+1,k)}\binom{n}{k-1}. \tag{4.2}$$

These then give interesting and useful statements about the g.c.d. and l.c.m. of consecutive binomial coefficients on the n^{th} row of the Pascal triangle. In principle, we may find relations for the g.c.d. and l.c.m. of $\binom{n}{k}$ and $\binom{m}{j}$.

REFERENCES

1. L. E. Dickson. *History of the Theory of Numbers*. Vol. 1. Washington, D.C.: Carnegie Institution, 1919; rpt. New York: Chelsea, 1952.
2. Hugh M. Edgar. "The Least Common Multiple of Some Binomial Coefficients." *The Fibonacci Quarterly* **24.4** (1986):310-12.
3. H. W. Gould. Problem 5777*. *Amer. Math. Monthly* **78** (1971):202.
4. H. W. Gould. "A New Primality Criterion of Mann and Shanks and Its Relation to a Theorem of Hermite with Extension to Fibonomials." *The Fibonacci Quarterly* **10.4** (1972): 355-72.
5. H. W. Gould. "A New Greatest Common Divisor Property of the Binomial Coefficients." *The Fibonacci Quarterly* **10.6** (1972):579-84, 628.
6. H. W. Gould. "Generalization of Hermite's Divisibility Theorems and the Mann-Shanks Primality Criterion for s-Fibonomial Arrays." *The Fibonacci Quarterly* **12.2** (1974):157-66.
7. H. W. Gould & W. E. Greig. "A Lucas Triangle Primality Criterion Dual to that of Mann-Shanks." *The Fibonacci Quarterly* **23.1** (1985):66-69.
8. H. Gupta. "On a Problem in Parity." *Indian J. Math.* **11** (1969):157-63.
9. C. Hermite. Problems 257-258. *Jour. de math. Speciales* (1889):19-22; (1891):70. Solutions given by E. Catalan.
10. C. Hermite & T. J. Stieltjes. *Correspondance d'Hermite et de Stieltjes* **1** (1905):415-16. Letter of 17 April 1889.
11. Henry B. Mann & Daniel Shanks. "A Necessary and Sufficient Condition for Primality and Its Source." *J. Comb. Theory*, Ser. Z, **13** (1972):131-34.
12. G. B. Mathews. *Math. Quest. Educ. Times* **52** (1890):63.
13. Glenn N. Michael. "A New Proof of an Old Property." *The Fibonacci Quarterly* **2.1** (1964): 57-58.
14. G. Ricci. "Sui coefficienti binomali e polinomiali: Una dimostrazione del teorema di Staudt-Clausen sui numeri di Bernoulli." *Gior. Mat. Battaglini* **69** (1931):9-12.
15. Theodor Schönemann. "Theorie der symmetrischen Functionen der Wurzeln einer Gleichung: Allgemeine Sätze über Congruenzen nebst einigen Anwendungen derselben." *J. Reine Angew. Math.* **19** (1839):231-43, 289-308.
16. Marlow Sholander. "Least Common Multiples and Highest Common Factors." *Amer. Math. Monthly* **68** (1961):984.
17. David Singmaster. "Notes on Binomial Coefficients: I; II; III." *J. London Math. Soc.*, Series 2, **8** (1974):545-48; 549-54; 555-60.
18. David Singmaster. "Notes on Binomial Coefficients: IV—Proof of a Conjecture of Gould on the G.C.D.'s of Two Triples of Binomial Coefficients." *The Fibonacci Quarterly* **11.3** (1973):282-84.
19. David Singmaster. "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers." Unpublished 47-page manuscript, January 1973.

20. Problem E 2686. *Amer. Math. Monthly* **84** (1977):820; Solution, **89** (1979):131. (Posed by Peter L. Montgomery.)
21. Problem E 3431. *Amer. Math. Monthly* **98** (1991):264; Editorial comment, **99** (1992):877, citing Problem E 2686 and a related paper of M. Nair. (Posed by Jeffrey Shallit.)
22. M. Nair. "On Chebyshev-Type Inequalities for Primes." *Amer. Math. Monthly* **89** (1982): 126-29.

AMS Classification Numbers: 11A05, 11B39, 05A10

Announcement

SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

July 14-July 19, 1996

INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT GRAZ
STEYRERGASSE 30
A-8010 GRAZ, AUSTRIA

LOCAL COMMITTEE

Robert Tichy, Chairman
Helmut Prodinger, Co-chairman
Peter Grabner
Peter Kirschenhofer

INTERNATIONAL COMMITTEE

A. F. Horadam (Australia), Co-chair
A. N. Philippou (Cyprus), Co-chair
G. E. Bergum (U.S.A.)
P. Filipponi (Italy)
H. Harborth (Germany)
Y. Horibe (Japan)
M. Johnson (U.S.A.)
P. Kiss (Hungary)
G. M. Phillips (Scotland)
J. Turner (New Zealand)
M. E. Waddill (U.S.A.)

LOCAL INFORMATION

For information on local housing, food, tours, etc., please contact:

PROFESSOR DR. ROBERT F. TICHY
INSTITUT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT GRAZ
STEYRERGASSE 30
A-8010 GRAZ, AUSTRIA

Call for Papers

The SEVENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Technische Universität Graz from July 14 to July 19, 1996. This conference will be sponsored jointly by the Fibonacci Association and Technische Universität Graz.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts and manuscripts should be sent in duplicate following the guidelines for submission of articles found on the inside front cover of any recent issue of *The Fibonacci Quarterly* to:

PROFESSOR GERALD E. BERGUM
THE FIBONACCI QUARTERLY
COMPUTER SCIENCE DEPARTMENT
BOX 2201
SOUTH DAKOTA STATE UNIVERSITY
BROOKINGS, SD 57007-1596

or

PROFESSOR F. T. HOWARD
DEPARTMENT OF MATH. & COMP. SCI.
BOX 7388 REYNOLDA STATION
WAKE FOREST UNIVERSITY
WINSTON-SALEM, NC 27109
