REAL PELL AND PELL-LUCAS NUMBERS WITH REAL SUBSCRIPTS

Alwyn F. Horadam
University of New England, Armidale, NSW 2351, Australia

Piero Filipponi
Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142, Rome, Italy
(*Submitted February 1994*)

1. INTRODUCTION AND GENERALITIES

The aim of this note is to extend the ideas explored in [3] to Pell numbers P_n and Pell-Lucas numbers Q_n.

More precisely, we shall parallel the arguments of [3] (the contents of which the reader is assumed to be aware of) to obtain expressions for both Pell numbers P_x and Pell-Lucas numbers Q_x which are *real* when the subscript $x \geq 0$ is a real quantity. Of course, these numbers (or better, functions) and the usual Pell numbers and Pell-Lucas numbers coincide when $x = n$ is an integer. It will be shown that P_x and Q_x enjoy some of the main properties of P_n and Q_n.

For the convenience of the reader, let us recall the Binet forms for Pell and Pell-Lucas numbers and some identities involving them. These are (e.g., see [1], [5])

$$P_n = (\alpha^n - \beta^n)/\sqrt{8} \quad \text{(Binet form)}, \tag{1.1}$$

$$Q_n = \alpha^n + \beta^n \quad \text{(Binet form)}, \tag{1.2}$$

where

$$\alpha = -1/\beta = 2 - \beta = 1 + \sqrt{2}, \tag{1.3}$$

$$P_{n+2} = 2P_{n+1} + P_n \quad [P_0 = 0, P_1 = 1] \quad \text{(recurrence relation)}, \tag{1.4}$$

$$Q_{n+2} = 2Q_{n+1} + Q_n \quad [Q_0 = Q_1 = 2] \quad \text{(recurrence relation)}, \tag{1.5}$$

$$Q_n = P_{n-1} + P_{n+1}, \tag{1.6}$$

$$P_n Q_n = P_{2n}, \tag{1.7}$$

$$P_{n-1}P_{n+1} = P_n^2 + (-1)^n \quad \text{(Simson formula analogue)}, \tag{1.8}$$

and

$$8P_n^2 = Q_n^2 - 4(-1)^n. \tag{1.9}$$

In section 2 the *exponential* representations for P_x and Q_x are defined for all x and coincide with P_n and Q_n, respectively, when n is an integer. In section 3 the *polynomial-exponential* representation for P_x is defined only for $x \geq 0$ and coincides with P_n when n is a nonnegative integer, whereas the *polynomial-exponential* representation for Q_x is defined only for $x > 0$ and coincides with Q_n when n is a positive integer. In both sections some properties of these numbers are established. Finally, the application of a useful idea [7] is discussed briefly in section 4. It must be noted that, despite the fact that *the numbers defined in sections 2-4 coincide only when $x = n$ is an integer*, *they are denoted by the same symbol.* Nevertheless, no misunderstanding can arise since each definition applies only to the appropriate section. The notation

$\lambda(x)$, the greatest integer not exceeding x,
$\phi(x) = x - \lambda(x)$, the fractional part of x,

will be used, and the following properties of $\lambda(x)$ will be taken into account throughout the proofs:

$$\lambda[(x \pm 1)/2] = \lambda(x/2) \pm [1 \mp (-1)^{\lambda(x)}]/2, \tag{1.10}$$

$$\lambda[(x-2)/2] = \lambda(x/2) - 1, \tag{1.11}$$

$$2\lambda(x/2) = \lambda(x) - [1-(-1)^{\lambda(x)}]/2, \tag{1.12}$$

$$\lambda(-x) = -\lambda(x) - 1 \text{ [i.e., } \lambda(x) + \lambda(-x) = -1\text{], if } \phi(x) > 0. \tag{1.13}$$

The proofs of (1.10)-(1.13) are not difficult but they are very lengthy and tedious. They are left to the perseverance of the reader. Further, the conventions

$$\binom{x}{-k} = 0, \text{ if } k \geq 1 \text{ is an integer ([2], p. 48)} \tag{1.14}$$

and

$$\sum_{i=a}^{b} f(i) = 0, \text{ if } b < a \tag{1.15}$$

will be assumed.

2. EXPONENTIAL REPRESENTATION OF P_x AND Q_x

Keeping the Binet forms (1.1) and (1.2), and the definitions (2.13) and (2.14) of [3] in mind, leads us to define

$$P_x = [\alpha^x - (-1)^{\lambda(x)}\alpha^{-x}]/\sqrt{8} \tag{2.1}$$

and

$$Q_x = \alpha^x + (-1)^{\lambda(x)}\alpha^{-x}. \tag{2.2}$$

As an illustration, the behavior of P_x vs x is shown in Figure 1 for $0 \leq x \leq 8$.

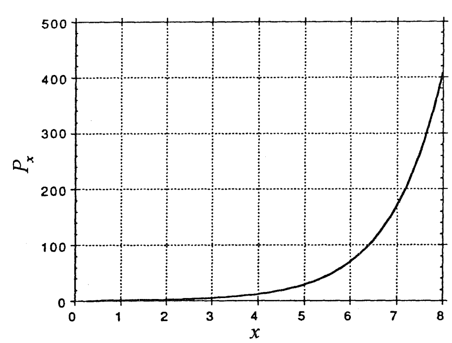

FIGURE 1. Behavior of P_x vs x for $0 \leq x \leq 8$

The same function is plotted, within the interval $0.5 \leq x \leq 2.5$ in Figure 2, to reveal the (rapidly decreasing) discontinuities connected with the integral values of x which are due to the greatest integer function inherent in the definition (2.1).

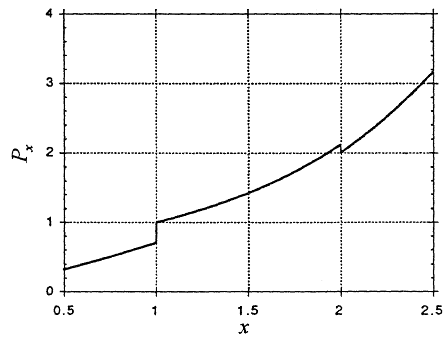

FIGURE 2. Graph of P_x vs x for $0.5 \le x \le 2.5$

2.1. Some Properties of P_x and Q_x

The numbers P_x and Q_x enjoy several properties of the usual Pell and Pell-Lucas numbers. For example, the identities (1.4)-(1.9) remain valid when n is replaced by x with only one exception. The exception is (1.7) which must be restated as follows.

Proposition 1:

$$P_xQ_x = \begin{cases} P_{2x}, & \text{if } \phi(x) < \tfrac{1}{2}, \\ P_{2x} - \alpha^{-2x}/\sqrt{2} = Q_{2x}/\sqrt{8}, & \text{if } \phi(x) \ge \tfrac{1}{2}. \end{cases}$$

This will be proved later. Moreover, it must be noted that the quantity $(-1)^n$ has to be replaced by $(-1)^{\lambda(x)}$ in (1.8) and (1.9).

The evaluation of finite sums analogous to those considered in [3] gives the results

$$\sum_{k=0}^{n} T_{x+k} = \frac{1}{2}(T_{n+1+x} + T_{n+x} - T_x - T_{x-1}), \tag{2.3}$$

where T stands separately for P and Q, and

$$\sum_{k=0}^{n-1} P_{k/n} = \frac{P_{(n-1)/n} + P_{1/n} - 1/\sqrt{2}}{2 - Q_{1/n}} \quad (n \ge 2), \tag{2.4}$$

$$\sum_{k=0}^{n-1} Q_{k/n} = \frac{Q_{(n-1)/n} - Q_{1/n} - 2(\sqrt{2}-1)}{2 - Q_{1/n}} \quad (n \ge 2). \tag{2.5}$$

The identities (2.3)-(2.5) can be proved by using (2.1), (2.2), and the geometric series formula.

The extension of P_x and Q_x to negative values of the subscript x can be obtained by replacing x by $-x$ in the definitions (2.1) and (2.2), and by taking (1.13) into account. To our great surprise, some simple calculations led to the following unexpected results

$$\begin{cases} P_{-x} = (-1)^{\lambda(x)} Q_x/\sqrt{8} & (2.6) \\ Q_{-x} = (-1)^{\lambda(x)+1} P_x\sqrt{8} & (2.7) \end{cases} \quad \text{for } \phi(x) > 0,$$

which hold whenever x is not an integer. In spite of the unexpectedness of expessions (2.6) and (2.7), the numbers P_{-x} and Q_{-x} preserve many properties of $P_{-n}=(-1)^{n+1}P_n$ and $Q_{-n}=(-1)^nQ_n$. For example, the identity

$$P_{-x}Q_{-x}=-P_xQ_x \quad \text{(see Proposition 1)} \tag{2.8}$$

holds whatever the nature of x.

2.2. Some Detailed Proofs

For space reasons, only a few among the properties stated in section 2.1 will be proved in detail. It is worth mentioning that the following equalities involving the quantity α [see (1.3)] are to be used in the proofs of (1.4)-(1.6):

$$2\alpha+1=\alpha^2, \tag{2.9}$$

$$1-2\alpha^{-1}=\alpha^{-2}, \tag{2.10}$$

$$1+\alpha^2=\alpha\sqrt{8}. \tag{2.11}$$

Proof of (1.5) (for n replaced by x): By (2.2),

$$\begin{aligned}
2Q_{x+1}+Q_x &= 2[\alpha^{x+1}+(-1)^{\lambda(x+1)}\alpha^{-x-1}]+\alpha^x+(-1)^{\lambda(x)}\alpha^{-x}\\
&= \alpha^x(2\alpha+1)+(-1)^{\lambda(x)}\alpha^{-x}(1-2\alpha^{-1}) \quad [\text{since } \lambda(x+k)=\lambda(x)+k,\ k \text{ an integer})\\
&= \alpha^{x+2}+(-1)^{\lambda(x)}\alpha^{-(x+2)} \quad [\text{by (2.9), (2.10)}]\\
&= \alpha^{x+2}+(-1)^{\lambda(x)+2}\alpha^{-(x+2)}\\
&= \alpha^{x+2}+(-1)^{\lambda(x+2)}\alpha^{-(x+2)}\\
&= Q_{x+2} \quad [\text{by (2.2)}]. \quad \text{Q.E.D.}
\end{aligned}$$

Proof of (1.8) (for n replaced by x): By (2.1),

$$\begin{aligned}
P_{x-1}P_{x+1}-P_x^2 &= ([\alpha^{x-1}+(-1)^{\lambda(x)}\alpha^{-x+1}][\alpha^{x+1}+(-1)^{\lambda(x)}\alpha^{-x-1}]-[\alpha^x-(-1)^{\lambda(x)}\alpha^{-x}]^2)/8\\
&= ([\alpha^{2x}+\alpha^{-2x}+(-1)^{\lambda(x)}(\alpha^2+\alpha^{-2})]-[\alpha^{2x}+\alpha^{-2x}-2(-1)^{\lambda(x)}])/8\\
&= (-1)^{\lambda(x)}(\alpha^2+\alpha^{-2}-2)/8\\
&= (-1)^{\lambda(x)}(\alpha-\alpha^{-1})^2/8\\
&= (-1)^{\lambda(x)} \quad [\text{since } \alpha-\alpha^{-1}=2\sqrt{2},\ \text{by (1.3)}]. \quad \text{Q.E.D.}
\end{aligned}$$

Proof of Proposition 1: By (2.1) and (2.2),

$$\begin{aligned}
P_xQ_x-P_{2x} &= (\alpha^{2x}-\alpha^{-2x})/\sqrt{8}-[\alpha^{2x}-(-1)^{\lambda(2x)}\alpha^{-2x}]/\sqrt{8}\\
&= \alpha^{-2x}[(-1)^{\lambda(2x)}-1]/\sqrt{8}\\
&= \begin{cases} 0, & \text{if } \lambda(2x) \text{ is even } \left[\text{i.e., if } \phi(x)<\tfrac{1}{2}\right],\\ -\alpha^{-2x}/\sqrt{2}, & \text{if } \lambda(2x) \text{ is odd } \left[\text{i.e., if } \phi(x)\ge\tfrac{1}{2}\right]. \end{cases} \quad \text{Q.E.D.}
\end{aligned}$$

3. POLYNOMIAL-EXPONENTIAL REPRESENTATION OF P_x AND Q_x

Keeping the definitions (1.6) and (1.7) of [4] and the definitions (3.4) and (3.5) of [3] in mind, leads us to define

$$P_x = \sum_{j=0}^{\lambda[(x-1)/2]} \binom{x-1-j}{j} 2^{x-1-2j} \quad (x \geq 0) \tag{3.1}$$

and

$$Q_x = \sum_{j=0}^{\lambda(x/2)} \frac{x}{x+j} \binom{x-j}{j} 2^{x-2j} \quad (x > 0). \tag{3.2}$$

Observe that the binomial coefficient defined as

$$\binom{x}{0} = 1, \ \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} \quad (k \geq 1, \text{ an integer}) \tag{3.3}$$

makes sense ([2], p. 48) also if x is any real quantity. Moreover, observe that

(i) for $x = 0$, the expression (3.2) gives the indeterminate form 0/0 so that $Q_0 = 2$ cannot be defined by (3.2),

(ii) by (1.13) and (1.15), we see that the expression (3.1) allows us to get $P_0 = 0$, and the extension to negative values of x yields $P_{-x} = Q_{-x} = 0$.

As an illustration, we show the first few values of P_x and Q_x. They are

$$P_x = 0 \quad (0 \leq x < 1),$$
$$P_x = 2^{x-1} \quad (1 \leq x < 3),$$
$$P_x = 2^{x-3}(x+2) \quad (3 \leq x < 5),$$
$$P_x = 2^{x-6}(x^2+x+28) \quad (5 \leq x < 7)$$
$$P_x = 2^{x-8}\left(\frac{1}{3}x^3 - x^2 + \frac{86}{3}x + 72\right) \quad (7 \leq x < 9)$$
$$P_x = 2^{x-11}\left(\frac{1}{6}x^4 - \frac{5}{3}x^3 + \frac{203}{6}x^2 + \frac{155}{3}x + 856\right) \quad (9 \leq x < 11),$$

and

$$Q_x = 2^x \quad (0 < x < 2),$$
$$Q_x = 2^{x-2}(x+4) \quad (2 \leq x < 4),$$
$$Q_x = 2^{x-5}(x^2+5x+32) \quad (4 \leq x < 6),$$
$$Q_x = 2^{x-7}\left(\frac{1}{3}x^3 + x^2 + \frac{80}{3}x + 128\right) \quad (6 \leq x < 8),$$
$$Q_x = 2^{x-10}\left(\frac{1}{6}x^4 - \frac{1}{3}x^3 + \frac{155}{6}x^2 + \frac{535}{3}x + 1024\right) \quad (8 \leq x < 10).$$

The behavior of P_x vs x is shown in Figure 3 for $0 \leq x \leq 5.5$.

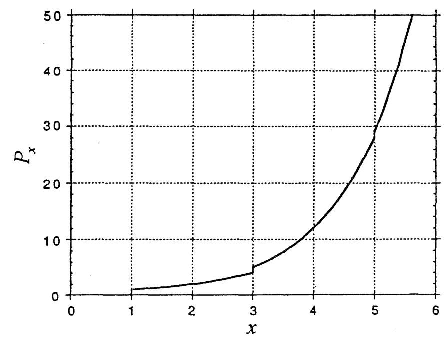

FIGURE 3. Behavior of P_x vs x for $0 \le x \le 5.5$

3.1. Some Properties of P_x and Q_x

Proposition 2:

$$2P_{x+1}+P_x=\begin{cases} P_{x+2}, & \text{if } \lambda(x) \text{ is even,}\\ P_{x+2}-\binom{x-\lambda(x/2)-1}{\lambda(x/2)+1}2^{\phi(x)}, & \text{if } \lambda(x) \text{ is odd.}\end{cases}$$

Proposition 3:

$$2Q_{x+1}+Q_x=\begin{cases} Q_{x+2}-\dfrac{x+1}{x-\lambda(x/2)}\binom{x-\lambda(x/2)}{\lambda(x/2)+1}2^{\phi(x)}, & \text{if } \lambda(x) \text{ is even,}\\ Q_{x+2}, & \text{if } \lambda(x) \text{ is odd.}\end{cases}$$

Proposition 4: $P_{x+1}+P_{x-1}=Q_x$.

3.2. Proofs

Proof of Proposition 2:

Case 1: $\underline{\lambda(x) \text{ even}}$. By (3.1) and (1.10), write

$$\begin{aligned} P_x+2P_{x+1} &= \sum_{j=0}^{\lambda(x/2)-1}\binom{x-1-j}{j}2^{x-1-2j}+\sum_{j=0}^{\lambda(x/2)}\binom{x-j}{j}2^{x+1-2j}\\ &= \sum_{j=1}^{\lambda(x/2)}\binom{x-j}{j-1}2^{x+1-2j}+\sum_{j=0}^{\lambda(x/2)}\binom{x-j}{j}2^{x+1-2j}. \end{aligned}$$

Taking (1.14) and (1.10) into account and using the basic recurrence ([8], p. 1) for the binomial coefficients (which holds also when the upper argument is not an integer) yields

$$\begin{aligned} P_x+2P_{x+1} &= \sum_{j=0}^{\lambda(x/2)}\left[\binom{x-j}{j-1}+\binom{x-j}{j}\right]2^{x+1-2j} = \sum_{j=0}^{\lambda(x/2)}\binom{x+1-j}{j}2^{x+1-2j}\\ &= \sum_{j=0}^{\lambda[(x+1)/2]}\binom{x+1-j}{j}2^{x+1-2j} = P_{x+2}. \end{aligned}$$

Case 2: $\lambda(x)$ odd. By (3.1), (1.10), and (1.12), write

$$P_x + 2P_{x+1} = \sum_{j=0}^{\lambda(x/2)} \binom{x-1-j}{j} 2^{x-1-2j} + \sum_{j=0}^{\lambda(x/2)} \binom{x-j}{j} 2^{x+1-2j}$$

$$= \sum_{j=1}^{\lambda(x/2)+1} \binom{x-j}{j-1} 2^{x+1-2j} + \sum_{j=0}^{\lambda(x/2)} \binom{x-j}{j} 2^{x+1-2j}$$

$$= \sum_{j=0}^{\lambda(x/2)+1} \binom{x-j}{j-1} 2^{x+1-2j} + \sum_{j=0}^{\lambda(x/2)+1} \binom{x-j}{j} 2^{x+1-2j} - X, \tag{3.4}$$

where

$$X = \binom{x-\lambda(x/2)-1}{\lambda(x/2)+1} 2^{x+1-2\lambda(x/2)-2} = \binom{x-\lambda(x/2)-1}{\lambda(x/2)+1} 2^{x-\lambda(x)}. \tag{3.5}$$

By (1.10), (3.5), and the basic recurrence for the binomial coefficients, expression (3.4) can be rewritten as

$$P_x + 2P_{x+1} = \sum_{j=0}^{\lambda[(x+1)/2]} \binom{x+1-j}{j} 2^{x+1-2j} - X = P_{x+2} - X.$$

The proposition follows, since $\phi(x) = x - \lambda(x)$. Q.E.D.

Note: (1) Since the upper argument of the binomial coefficient in (3.5) is less than the lower one, $X = 0$ whenever $x \geq 1$ is an (odd) integer, giving $2P_{x+1} + P_x = P_{x+2}$.

(2) Proposition 3 may be proved in a way similar to Proposition 2.

Proof of Proposition 4: First, by (3.1), (3.2), and the binomial identity available in ([8], p. 64), write

$$Q_x = \sum_{j=0}^{\lambda(x/2)} \left[\binom{x-j}{j} + \binom{x-1-j}{j-1}\right] 2^{x-2j} = P_{x+1} + \sum_{j=0}^{\lambda(x/2)} \binom{x-1-j}{j-1} 2^{x-2j}$$

$$= P_{x+1} + \sum_{j=-1}^{\lambda(x/2)-1} \binom{x-2-j}{j} 2^{x-2-2j}. \tag{3.6}$$

Then, use (1.14), (1.11), and (3.1) to rewrite (3.6) as

$$Q_x = P_{x+1} + \sum_{j=0}^{\lambda[(x-2)/2]} \binom{x-2-j}{j} 2^{x-2-2j} = P_{x+1} + P_{x-1}. \quad \text{Q.E.D.}$$

4. CONCLUDING REMARKS

In this note, definitions have been proposed for Pell numbers P_x and Pell-Lucas numbers Q_x which are real when the subscript x is real. We feel that this particular study might be concluded suitably by observing that the idea explored in [7] applies beautifully to the afore-said numbers (see also [6]). In fact, following [7], we can define

$$P_x = [\alpha^x - \cos(\pi x)\alpha^{-x}] / \sqrt{8} \tag{4.1}$$

and

$$Q_x = \alpha^x + \cos(\pi x)\alpha^{-x}. \tag{4.2}$$

The numbers P_x and Q_x defined in this way and the usual Pell and Pell-Lucas numbers obviously coincide when x is an integer. Moreover, their behavior vs x does not present any discontinuity, as shown in Figure 4 in the case of Q_x.

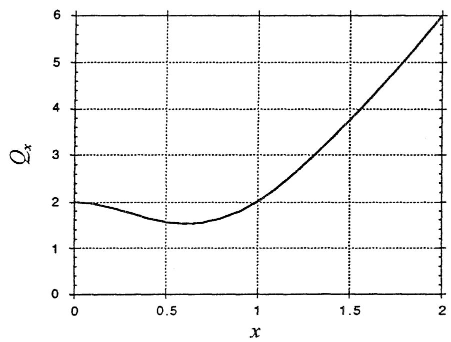

FIGURE 4. Behavior of Q_x vs x for $0 \le x \le 2$

Some properties of these numbers are reported in the sequel. Their proofs are left as an exercise for the interested reader. It should be noted that (4.1) and (4.2) occur in [6] as x coordinates of points on Pell and Pell-Lucas curves. Both x- and y-coordinates for these curves were obtained independently of [7] as special cases of coordinates for a system of more general curves [6].

The identities (1.4)-(1.6) remain valid for P_x and Q_x, whereas the identity (1.7) does not. More precisely, we have

$$P_x Q_x = P_{2x} - [\sin^2(\pi x)\alpha^{-2x}]/\sqrt{8}. \tag{4.3}$$

Moreover, the analogue of (1.8) is

$$P_{x-1}P_{x+1} - P_x^2 = \cos(\pi x). \tag{4.4}$$

The extensions of (4.1) and (4.2) to negative values of x lead to

$$P_{-x} = \begin{cases} [\sin^2(\pi x)\alpha^x/\sqrt{8} - P_x]/\cos(\pi x), & \text{if } \phi(x) \ne \frac{1}{2}, \\ P_x^{-1}/8, & \text{if } \phi(x) = \frac{1}{2}, \end{cases} \tag{4.5}$$

and

$$Q_{-x} = \begin{cases} [Q_x - \sin^2(\pi x)\alpha^x]/\cos(\pi x), & \text{if } \phi(x) \ne \frac{1}{2}, \\ Q_x^{-1}, & \text{if } \phi(x) = \frac{1}{2}. \end{cases} \tag{4.6}$$

Since the reader might find some difficulty in deriving (4.5) and (4.6),we give a sketch of the proof of (4.5).

Proof of (4.5) (a sketch): Replace x by $-x$ in (4.1), thus getting

$$P_{-x} = [\alpha^{-x} - \cos(\pi x)\alpha^x]/\sqrt{8} \quad [\text{since } \cos(-y) = \cos y]. \tag{4.7}$$

If $\phi(x) = \frac{1}{2}$, then $\cos(\pi x) = 0$ so that $P_x = \alpha^x / \sqrt{8}$ [see (4.1)], and $P_x^{-1} = \sqrt{8}\alpha^{-x} = 8\alpha^{-x} / \sqrt{8} = 8P_{-x}$ [see (4.7)]. If $\phi(x) \neq \frac{1}{2}$ [i.e., $\cos(\pi x) \neq 0$], multiply both sides of (4.7) by $\cos(\pi x)$, and use the identity $\cos^2 y = 1 - \sin^2 y$ to obtain the right-hand side of (4.5). Q.E.D.

The proof of (4.6) is similar. Observe that (4.5) and (4.6) do not satisfy the analogue of (2.8) for $\phi(x) > 0$. In particular, when $\phi(x) = \frac{1}{2}$ (i.e., $x = n + \frac{1}{2}$), we have

$$P_x Q_x + P_{-x} Q_{-x} = P_{2n+1}. \tag{4.8}$$

Furthermore, the identity (2.3) remains valid for P_x and Q_x, whereas an attempt to find the identities corresponding to (2.4) and (2.5) required a great amount of calculations involving the use of Euler formulas for circular functions and the geometric series formula, and produced a couple of very unpleasant expressions. As an illustration, we exhibit the second one. This is

$$\sum_{k=0}^{n-1} Q_{k/n} = \frac{\sqrt{2}[\sqrt{8}P_{1/n} + \alpha^{-2/n} - \cos(\pi / n)]}{Q_{1/n} + \alpha^{-1/n} - \alpha^{-2/n} + (\alpha^{-1/n} - 2)\cos(\pi / n) - 1}. \tag{4.9}$$

The closed-form expression of the analogous sum

$$\sum_{k=0}^{n-1} Q_{k/n}\alpha^{k/n} = \frac{\alpha^2 - 1}{\alpha^{2/n} - 1} + 1 \tag{4.10}$$

is much simpler even though perhaps less interesting.

ACKNOWLEDGMENT

The contribution of the second author has been given within the framework of an agreement between the Italian PT Administration and the Fondazione Ugo Bordoni.

REFERENCES

1. M. Bicknell. "A Primer on the Pell Sequence and Related Sequences." *The Fibonacci Quarterly* **13.4** (1975):345-49.
2. W. Feller. *An Introduction to Probability Theory and Its Applications*. Vol I. 2nd ed. New York: Wiley, 1957.
3. P. Filipponi. "Real Fibonacci and Lucas Numbers with Real Subscripts." *The Fibonacci Quarterly* **31.4** (1993):307-14.
4. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers*, **4:**99-108. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecth: Kluwer, 1991.
5. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
6. A. F. Horadam. "Jacobsthal and Pell Curves." *The Fibonacci Quarterly* **26.1** (1988):77-83.
7. F. D. Parker. "A Fibonacci Function." *The Fibonacci Quarterly* **6.1** (1968):1-2.
8. J. Riordan. *Combinatorial Identities*. New York: Wiley, 1968.

AMS Classification Numbers: 11B65, 33B10, 11B39

❖❖❖