NOTES ON A CONJECTURE OF SINGMASTER

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1. INTRODUCTION

Let $\{a_i\}_{i=1}^n$ be a sequence of positive integers in nondecreasing order. Following Guy [1], $\{a_i\}$ is a sum=product sequence of size n if $\sum_{i=1}^n a_i = \prod_{i=1}^n a_i$. For example, it is easily shown that $\{2,2\}$, $\{1,2,3\}$, and $\{1,1,2,4\}$ are the only sum=product sequences having sizes 2, 3, and 4, respectively. Let $N(n)$ denote the number of different sum=product sequences of size n. Basing his research on various numerical data obtained by computer, David Singmaster has made some conjectures about $N(n)$. These conjectures were proposed during the closing session of the Fifth International Conference on Fibonacci Numbers and Their Applications (St. Andrews, Scotland, 1992); namely, $N(n)>1$ for $n>444$, $N(n)>2$ for $n>6324$, and $N(n)>3$ for $n>11874$. The most attractive conjecture is the statement that $N(n)\to\infty$ as $n\to\infty$.

The object of this note is twofold. First, we give an explicit expression for $N(n)$. Second, we investigate an extended conjecture for the number $N(n,k)$ of different $(\text{sum})^k$=product sequences of size n $(n>k\geq 2)$. Then our extended conjecture is the assertion that $N(n)=\infty$ for $n>k\geq 2$. We prove this extended conjecture.

2. AN EXPRESSION FOR $N(n)$

As usual, denote by $[x]$ the integer part of $x>0$. Let $r_k(n)$ denote the number of different ordered solutions of the Diophantine equation, with $2\leq x_1\leq x_2\leq\cdots\leq x_k$,

$$\prod_{i=1}^k x_i - \sum_{i=1}^k x_i = n-k \quad (n>k\geq 2). \tag{1}$$

Moreover, we introduce a unit function $I\{x\}$ defined for rational numbers x by the following:

$$I\{x\} = \begin{cases} 1 & \text{if } x \text{ is a nonnegative integer,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Proposition 1: Let $d(n)$ be the divisor function representing the number of divisors of n. For $n>3$, we have

$$N(n) = \left[\frac{1}{2}(d(n-1)+1)\right] + \sum_{k=3}^m r_k(n), \tag{3}$$

where $m=[\log_2 n]+2$ and $r_k(n)$ may be expressed in the form

$$r_k(n) = \sum_{2 \le x_1 \le \cdots \le x_{k-1}} I\left\{\frac{n-k+x_1+\cdots+x_{k-1}}{(x_1\cdots x_{k-1})-1} - x_{k-1}\right\}, \tag{4}$$

the summation being taken over all integers x_i with $2 \le x_1 \le \cdots \le x_{k-1}$.

Proof: Notice that for any ordered solution $(x_1, \ldots, x_k)$ of (1) with $k \ge 2$ and $x_1 \ge 2$ we may write

$$\sum_{i=1}^{n-k} 1 + \sum_{i=1}^{k} x_i = \left(\prod_{i=1}^{n-k} 1\right) \prod_{i=1}^{k} x_i$$

so that it yields a sum=product sequence of size n. Thus, $N(n)$ may be expressed in the form

$$N(n) = r_2(n) + r_3(n) + \cdots.$$

Here the first term $r_2(n)$ just represents the number of ordered solutions of the equation $x_1x_2 - x_1 - x_2 = n-2$, which may be rewritten as $(x_1-1)(x_2-1) = n-1$. Since the number of divisors of $(n-1)$ is given by $d(n-1)$, it is clear that the number of distinct pairs (x_1-1, x_2-1) with $x_1 \le x_2$ should be equal to $\left[\frac{1}{2}(d(n-1)+1)\right]$, which is precisely the first term of (3).

To show that $m = [\log_2 n] + 2$, it suffices to determine the largest possible k such that equation (1) with $n > k \ge 3$ may have integer solutions in $x_i \ge 2$. Now, by induction on k, it can be shown that the following inequality,

$$\prod_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i \ge \frac{1}{4}\prod_{i=1}^{k} x_i,$$

holds for all $x_i \ge 2$ and $k \ge 3$. (Here the routine induction proof is omitted.) Consequently, we may infer the following from (1):

$$\frac{1}{4}x_1x_2\cdots x_k \le n-k < n.$$

Clearly, the largest possible k, viz. $m = \max\{k\}$, may be obtained by setting all $x_i = 2$. Thus, we have $2^{m-2} < n$, and we obtain $m = [\log_2 n] + 2$.

Finally, let us show that $r_k(n)$ has the expression (4). As may be observed, one may solve (1) for x_k in terms of integers $x_i \ge 2$ $(i = 1, \ldots, k-1)$,

$$x_k = \left(n - k + \sum_{i=1}^{k-1} x_i\right) \Big/ \left(\prod_{i=1}^{k-1} x_i - 1\right).$$

Therefore, every ordered solution of (1) with $2 \le x_1 \le \cdots \le x_k$ just corresponds to the condition $I\{x_k - k_{k-1}\} = 1$ and vice versa. Consequently, the number $r_k(n)$ (with $n > k \ge 3$) can be expressed as the summation (4). □

As may be verified, (4) can be used in a straightforward manner to give the value $r_2(n) = \left[\frac{1}{2}(d(n-1)+1)\right]$. However, there seems to be no way to simplify the summation (4) for the general case $k \ge 3$, although for given n and k the sum can be found using a computer.

Corollary 1: $N(n) \ge \left[\frac{1}{2}(d(n-1)+1)\right]$ for $n \ge 3$.

Corollary 2: $\limsup_{n\to\infty} N(n) = \infty$.

Corollaries 1 and 2 were also observed by Singmaster and his coauthors (cf. their preprint [2]). The following simple examples are immediate consequences of the corollaries.

Example 1: For $m > 1$, we have $N(m^n + 1) \to \infty$ $(n \to \infty)$.

Example 2: If $\{p_n\}$ is the sequence of prime numbers, then we have $N(p_1 p_2 \cdots p_n + 1) \to \infty$ as $n \to \infty$.

3. THE EXTENDED CONJECTURE

Given n and k with $n > k \geq 2$. The so-called extended conjecture is the statement that the number of different solutions of the Diophantine equation

$$\left(\sum_{i=1}^{n} x_i\right)^k = \prod_{i=1}^{n} x_i$$

is infinite, namely, $N(n, k) = \infty$.

In what follows, we will prove the extended conjecture.

Theorem 1: For $n > k \geq 2$, the Diophantine equation

$$\left(\sum_{i=1}^{n} x_i\right)^k = \prod_{i=1}^{n} x_i \tag{5}$$

has infinitely many solutions, namely, $N(n, k) = \infty$.

We shall accomplish the proof using three lemmas.

Lemma 1: For given integers $m \geq 0$, $\lambda \geq 1$, and $r \geq 2$, if the equation

$$\left(m + \sum_{i=1}^{r} x_i\right)^2 = \lambda \prod_{i=1}^{r} x_i \tag{6}$$

has a solution, then it has infinitely many solutions.

Proof: For the simplest case $m = 0$ and $r = 2$, let the equation

$$(x_1 + x_2)^2 = \lambda x_1 x_2 \tag{7}$$

have a solution $(x_1, x_2) = (a_1, a_2)$. Without loss of generality, assume $\gcd(a_1, a_2) = 1$. Then (7) implies $a_1 | a_2^2, a_2 | a_1^2$, so that $a_1 = a_2 = 1$ and consequently $\lambda = 4$. Now, evidently, (7) has infinitely many solutions (x_1, x_2) with $x_1 = x_2$ and $\lambda = 4$.

Consider the general case $m > 0$ or $r > 2$. Now suppose (6) has a solution $A = (a_1, a_2, \ldots, a_r)$ with $a_1 \geq a_2 \geq \cdots \geq a_r$. We shall construct a solution $B = (b_1, b_2, \ldots, b_r)$ with $b_1 \geq b_2 \geq \cdots \geq b_r$ different from A as follows. Denote $\|A\| = \max_{1 \leq i \leq r} a_i = a_1$. Consider the quadratic equation in t:

$$\left(m + \sum_{i=1}^{r-1} a_i + t\right)^2 = \lambda\left(\prod_{i=1}^{r-1} a_i\right)t. \tag{8}$$

i.e.,

$$t^2+\left\{2\left(m+\sum_{i=1}^{r-1}a_i\right)-\lambda\prod_{i=1}^{r-1}a_i\right\}t+\left(m+\sum_{i=1}^{r-1}a_i\right)^2=0.$$

By supposition, (8) has a root $t_1=a_r$. Using the relations between the roots and coefficients (Vieta's theorem), we see that the second root is given by

$$t_2=\lambda\sum_{i=1}^{r-1}a_i-2\left(m+\sum_{i=1}^{r-1}a_i\right)-t_1=\left(m+\sum_{i=1}^{r-1}a_i\right)^2\Big/t_1. \tag{9}$$

From (9), we see that t_2 is an integer and, moreover,

$$t_2=\left(m+\sum_{i=1}^{r-1}a_i\right)^2\Big/a_r>a_1^2/a_r\geq a_1.$$

Now let us take $b_1=t_2, b_2=a_1, b_3=a_2, \ldots, b_r=a_{r-1}$. From (8), it is clear that $B=(b_1,b_2,\ldots,b_r)$ is a solution of (6) with $\|B\|=\max_i b_i=b_1=t_2>a_1=\|A\|$, i.e., $\|B\|>\|A\|$.

Generally, if (6) has a solution $x^{(0)}=(x_1^{(0)},x_2^{(0)},\ldots,x_r^{(0)})$ with $x_1^{(0)}\geq x_2^{(0)}\geq\cdots\geq x_r^{(0)}$, then the recursive algorithm

$$\begin{cases}x_1^{(j+1)}=\lambda\prod_{i=1}^{r-1}x_i^{(j)}-2\left(m+\sum_{i=1}^{r-1}x_i^{(j)}\right)-x_r^{(j)},\\ x_2^{(j+1)}=x_1^{(j)},\ x_3^{(j+1)}=x_i^{(j)},\ \ldots,\ x_r^{(j+1)}=x_{r-1}^{(j)},\end{cases} \tag{10}$$

will yield infinitely many solutions $x^{(j)}=(x_1^{(j)},x_2^{(j)},\ldots,x_r^{(j)})$, $j=0,1,2,\ldots$, such that $\|x^{(0)}\|<\|x^{(1)}\|<\|x^{(2)}\|<\cdots$. □

Lemma 2: Let $m>0$ and $r\geq 3$. Then the equation

$$\left(m+\sum_{i=1}^{r}x_i\right)^2=\prod_{i=1}^{r}x_i \tag{11}$$

has infinitely many solutions.

Proof: Equation (11) is a form of (6) with $\lambda=1$. Now (11) has a solution $x_1=5(m+r+2)$, $x_2=4(m+r+2)$, $x_3=5$, $x_i=1$, $i=4,5,\ldots,r$. In fact,

$$\left(m+\sum_{i=1}^{r}x_i\right)^2=(m+5(m+r+2)+4(m+r+2)+5+(r-3))^2=100(m+r+2)^2=\prod_{i=1}^{r}x_i.\ \square$$

Hence, Lemma 2 follows from Lemma 1.

In particular, taking $m=0$ in (11), we get

Corollary 3: $N(n,2)=\infty$, where $n\geq 3$.

Lemma 3: Let $m\geq 1, r\geq 3$. Then

$$m\left(\sum_{i=1}^{r}x_i\right)^2=\prod_{i=1}^{r}x_i \tag{12}$$

has infinitely many solutions.

Proof: For the case $r = 3$, the substitution $x_i - my_i$ $(i = 1, 2, 3)$ in (12) leads to

$$\left(\sum_{i=1}^{3} y_i\right)^2 = \prod_{i=1}^{3} y_i .$$

For the case $r > 3$, taking $x_r = m$, we find that (12) becomes

$$\left(m + \sum_{i=1}^{r-1} x_i\right)^2 = \prod_{i=1}^{r-1} x_i .$$

Hence, Lemma 3 is implied by Lemma 2. □

Proof of Theorem 1: It suffices to prove the theorem "$N(n, k) = \infty$" for the case $k \geq 3$. In (12), let us take

$$m = 2^{(k-2)(k+3)/2}, \quad r = n - k + 2.$$

We will now show that from every solution $A = (a_1, a_2, \dots, a_r)$ with $a_1 \leq a_2 \leq \cdots \leq a_r$ of (12) there can be constructed a solution $x = (x_1, \dots, x_n)$ of (9) by the following:

$$\begin{cases} x_i = a_1, \ 1 = 1, 2, \dots, r; \\ x_{r+j} = 2^{j-1} \sum_{i=1}^{r} a_i, \ j = 1, 2, \dots, k-2. \end{cases} \tag{13}$$

In fact we have, by computation:

$$\left(\sum_{i=1}^{n} x_i\right)^k = \left[\sum_{i=1}^{r} a_i + \sum_{j=1}^{k-2}\left(2^{j-1}\sum_{i=1}^{r} a_i\right)\right]^k = \left(1 + \sum_{j=1}^{k-2} 2^{j-1}\right)^k \left(\sum_{i=1}^{r} a_i\right)^k = 2^{k(k-2)}\left(\sum_{i=1}^{r} a_i\right)^k,$$

and

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{r} a_i \cdot \prod_{j=1}^{k-2}\left(2^{j-1}\sum_{i=1}^{r} a_i\right)^2 = m\left(\sum_{i=1}^{r} a_i\right)^2 \cdot 2^{(k-2)(k+3)/2}\left(\sum_{i=1}^{r} a_i\right)^{k-2} = 2^{k(k-2)}\left(\sum_{i=1}^{r} a_i\right)^k .$$

That is,

$$\left(\sum_{1}^{n} x_i\right)^k = \prod_{1}^{n} x_i .$$

Clearly, $x_1 \leq x_2 \leq \cdots \leq x_{r+k-2} = x_n$ so that $\|x\| > \|A\|$. The recursive algorithm (10) implies that $\{\|A\|\}$ is unbounded, so is $\{\|x\|\}$. Hence, (5) also has infinitely many solutions. □

Theorem 2: For $n \geq 2k \geq 4$, the Diophantine equation

$$\left(\sum_{i=1}^{n} x_i\right)^k = \prod_{i=1}^{m} x_i$$

has at least $p(k)$ distinct solutions $(x_i, \dots, x_n)$ which are contained in the simplex domain

$$0 < \sum_{i=1}^{n} x_i < (k+1)^{k+1} + (k+1)n \quad (x_i > 0),$$

where $p(k)$ is the partition function of k.

Proof: Every partition of k may be represented by the summation

$$k = \alpha_1 + \alpha_2 + \cdots + \alpha_v \quad (1 \le v \le k),$$

where α_i are positive integers such that $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_v$. Denote $n = m + k + 1$ with $m \ge k - 1$. Then, corresponding to each partition $(\alpha_1 + \alpha_2 + \cdots + \alpha_v)$ of k, one can construct a solution of the equation as follows:

$$\left\{\begin{array}{l} x_i = (k+1)^{\alpha_1} + (k+1)^{\alpha_2} + \cdots + (k+1)^{\alpha_v} + m - v + 1 \text{ for } i = 1, \ldots, k, \\ x_{k+1} = (k+1)^{\alpha_1}, x_{k+2} = (k+1)^{\alpha_2}, \ldots, x_{k+v} = (k+1)^{\alpha_v}; \\ x_j = 1 \text{ for } j = k + v + 1, \ldots, k + m + 1. \end{array}\right\}$$

In fact, it may be verified at once that

$$\begin{aligned} \left(\sum_{i=1}^{n} x_i\right)^k &= \left\{[k(k+1)^{\alpha_1} + \cdots + (k+1)^{\alpha_v} + m - v + 1] \cdot k + \sum_{i=1}^{v} (k+1)^{\alpha_i} + m - v + 1\right\}^k \\ &= [(k+1)^{\alpha_1} + \cdots + (k+1)^{\alpha_v} + m - v + 1]^k \cdot (k+1)^k \\ &= [(k+1)^{\alpha_1} + \cdots + (k+1)^{\alpha_v} + m - v + 1]^k \prod_{i=1}^{v} (k+1)^{\alpha_i} = \prod_{i=1}^{n} x_i. \end{aligned}$$

Evidently the solution constructed above satisfies the condition

$$\begin{aligned} \sum_{i=1}^{n} x_i &= [(k+1)^{\alpha_1} + \cdots + (k+1)^{\alpha_v} + m - v + 1](k+1) \\ &\le [(k+1)^k + m](k+1) < (k+1)^{k+1} + (k+1)n. \end{aligned}$$

Hence, all the $p(k)$ distinct solutions are contained in the simplex domain as mentioned in the theorem. □

Example 3: For $n = 10, k = 5$, the equation $\left(\sum_{i=1}^{10} x_i\right)^5 = x_1 x_2 \cdots x_{10}$ has as least $p(5) = 7$ different solutions contained in the interior of the region: $0 < x_1 + x_2 + \cdots + x_{10} < 6^6 + 60 \; (x_i \ge 0)$.

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