LINEAR RECURRENCES IN DIFFERENCE TRIANGLES

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INTRODUCTION

This paper arose from our interest in generalizing a problem in the February 1993 issue of *The Fibonacci Quarterly* proposed by Piero Filipponi:

> Write down the Pell sequence, defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Form a difference triangle by writing down the successive differences in rows below it. . . . Identify the pattern that emerges down the left side and prove that this pattern continues. [1]

We investigate properties of difference triangles in which the sequence of numbers in the top row satisfies a linear homogeneous recurrence with constant coefficients. These coefficients and the entries in the sequences are integers in the examples in this paper. In the proofs, we assume that we are working over any field containing the integers.

1. DEFINITIONS AND NOTATIONS

We represent difference triangles (e.g., Fig. 1a) in matrix form (Fig. 1b) and refer to their rows and columns (rather than to their diagonals). Let d^0 denote the top (0^{th}) row of a difference triangle and let $d^i, i > 0$, denote the i^{th} row. Similarly, let d_0 denote the left-most (0^{th}) column of a difference triangle and let d_j, $j > 0$, denote the j^{th} column. The same symbols also denote the corresponding sequences of numbers in the rows and columns.

Let d_j^i denote the element in the i^{th} row and the j^{th} column of the difference triangle (e.g., $d_2^1 = -2$ in Fig. 1b). The difference triangle itself may be considered as a double sequence $\{d_j^i\}, i \geq 0, j \geq 0$, which obeys

$$d_j^i = d_{j+1}^{i-1} - d_j^{i-1} \text{ for } i \geq 1, j \geq 0. \qquad \text{(A)}$$

If the top row of a difference triangle is given, then (A) will yield all the other rows recursively.

This paper deals with difference triangles whose top row satisfies a linear recurrence that is homogeneous and has constant coefficients (LRHCC). Such a recurrence can be characterized by a nonnegative integer k called the *order* of the recurrence, together with an ordered set of k constants $c_0, c_1, \ldots, c_{k-1}$. A sequence $\{a_i\}$ is said to *satisfy* this recurrence if the following equation holds for each $n \geq 0$:

$$a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n \tag{B}$$

If $k = 0$, this becomes $a_n = 0$ for $n \geq 0$.

1		2		7		5		9
	1		5		−2		4	
		4		−7		6		
			−11		13			
				24				

(1a)

1	2	7	5	9
1	5	−2	4	
4	−7	6		
−11	13			
24				

(1b)

FIGURE 1. Triangular form (1a) and matrix form (1b) of difference triangle arising from the sequence 1, 2, 7, 5, 9.

2. RECURRENCE RELATIONS

Theorem 1a: Let $\{d_j^i\}$ be a difference triangle, k a nonnegative integer, and $c_0, c_1, \ldots, c_k$ constants such that, for all nonnegative integers n,

$$c_k d_{n+k}^i + c_{k-1} d_{n+k-1}^i + \cdots + c_1 d_{n+1}^i + c_0 d_n^i = 0 \tag{C}$$

holds for $i = m$, where m is some nonnegative integer. Then (C) also holds for all $i > m$.

Proof: We suppose that (C) holds for $i = p$ and show that it holds for $i = p+1$. Subtracting (C) with $i = p$ with n unchanged from (C) with $i = p$ and with n replaced by $n+1$, we obtain

$$c_k(d_{n+1+k}^p - d_{n+k}^p) + c_{k-1}(d_{n+k}^p - d_{n+k-1}^p) + \cdots + c_1(d_{n+2}^p - d_{n+1}^p) + c_0(d_{n+1}^p - d_n^p) = 0$$

for $n \geq 0$.

Applying (A) to the parenthetical expressions, we get

$$c_k(d_{n+k}^{p+1}) + c_{k-1}(d_{n+k-1}^{p+1}) + \cdots + c_1(d_{n+1}^{p+1}) + c_0(d_n^{p+1}) = 0$$

for $n \geq 0$, which is equation (C) with $i = p+1$. Since (C) holds for $i = m$, it now follows by induction that it also holds for all $i > m$.

Corollary: If the top row of a difference triangle satisfies a given LRHCC, then every row of the triangle satisfies the same recurrence.

Theorem 1b Let $\{d_j^i\}$ be a difference triangle, k a nonnegative integer, and $b_0, b_1, \ldots, b_k$ constants such that, for all $m \geq 0$,

$$b_k d_j^{m+k} + b_{k-1} d_j^{m+k-1} + \cdots + b_1 d_j^{m+1} + b_0 d_j^m = 0$$

holds for $j = n$ where n is some nonnegative integer. Then it also holds for all $j > n$.

Proof: Rewriting (A) as

$$d_{j+1}^{i-1} = d_j^{i-1} + d_j^i \quad \text{for } i \geq 1,\ j \geq 0, \tag{A$'$}$$

we obtain an analogous proof for the columns as for the rows in Theorem 1a.

Corollary: If the 0^{th} (left-most) column of a difference triangle satisfies a given LRHCC, then every column of it satisfies the same recurrence.

Lemma 1a: $d_j^i = \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} d_{j+\ell-s}^{i-\ell}$ for $i \geq \ell \geq 0,\ j \geq 0$.

Proof: Iterate (A) to obtain

$$\begin{aligned} d_j^i &= d_{j+1}^{i-1} - d_j^{i-1} = (d_{j+2}^{i-2} - d_{j+1}^{i-2}) - (d_{j+1}^{i-2} - d_j^{i-2}) \\ &= d_{j+2}^{i-2} - 2d_{j+1}^{i-2} + d_j^{i-2} \quad \text{for } i \geq 2,\ j \geq 0. \end{aligned}$$

Continuing, we express an element of the difference triangle as a (linear) function of the elements in the row that is ℓ rows above it:

$$d_j^i = \binom{\ell}{0} d_{j+\ell}^{i-\ell} - \binom{\ell}{1} d_{j+\ell-1}^{i-\ell} + \cdots + (-1)^{\ell-1} \binom{\ell}{\ell-1} d_{j+1}^{i-\ell} + (-1)^{\ell} \binom{\ell}{\ell} d_j^{i-\ell} \quad \text{for } i \geq \ell \geq 0,\ j \geq 0.$$

Lemma 1b: $d_j^i = \sum_{s=0}^{\ell} \binom{\ell}{s} d_{j-\ell}^{i+s}$ for $i \geq 0,\ j \geq \ell \geq 0$.

Proof: Iteration of (A$'$) gives

$$d_j^i = \binom{\ell}{0} d_{j-\ell}^{i} + \binom{\ell}{1} d_{j-\ell}^{i+1} + \cdots + \binom{\ell}{\ell-1} d_{j-\ell}^{i+\ell-1} + \binom{\ell}{\ell} d_{j-\ell}^{i+\ell} \quad \text{for } j \geq \ell \geq 0,\ i \geq 0.$$

Lemmas 1a and 1b are extensions of results found in the literature (cf. Hartree [2], p. 38, and Lakshmikantham & Trigiante [3], p. 3).

Theorem 2: If the top row of a difference triangle satisfies a k^{th} order LRHCC, then the left-most column also satisfies some k^{th} order LRHCC.

Proof: The LRHCC of the top row may be written as $c_k d_{n+k}^0 + c_{k-1} d_{n+k-1}^0 + \cdots + c_1 d_{n+1}^0 + c_0 d_n^0 = 0$ for $n \geq 0$, where, in this case, $c_k = -1$. By Theorem 1a, we may replace the superscript 0 with any nonnegative integer i. Then setting n equal to 0, we obtain

$$[c_0, c_1, \ldots, c_{k-1}, c_k] \begin{bmatrix} d_0^i \\ d_1^i \\ \vdots \\ d_k^i \end{bmatrix} = 0 \quad \text{for } i \geq 0. \tag{D}$$

Now setting $\ell = j$ in Lemma 1b for $0 \leq j \leq k$, and using matrix notation, we can write the above column vector as

$$\begin{bmatrix} d_0^i \\ d_1^i \\ \vdots \\ d_k^i \end{bmatrix} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{1} & \cdots & \cdots & \binom{k}{k} \end{bmatrix} \begin{bmatrix} d_0^i \\ d_0^{i+1} \\ \vdots \\ d_0^{i+k} \end{bmatrix} \quad \text{for } i \geq 0. \tag{D$'$}$$

Substitution of (D′) into (D) yields

$$[c_0, c_1, \ldots, c_{k-1}, c_k] \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{1} & \cdots & \cdots & \binom{k}{k} \end{bmatrix} \begin{bmatrix} d_0^i \\ d_0^{i+1} \\ \vdots \\ d_0^{i+k} \end{bmatrix} = 0 \quad \text{for } i \geq 0.$$

Multiplying the first two matrices, we obtain $[b_0, b_1, b_2, \ldots, b_{k-1}, b_k]$, where

$$b_j = \sum_{\ell=j}^{k} \binom{\ell}{j} c_\ell \quad \text{for } j = 0, 1, \ldots, k, \tag{D$''$}$$

so that the equality

$$b_0 d_0^i + b_1 d_0^{i+1} + \cdots + b_k d_0^{i+k} = 0$$

holds for each $i \geq 0$. Since the b's do not depend on i, and since $b_k = -1$, the left-most column satisfies a k^{th}-order LRHCC. Note that, generally, this LRHCC is not the same as for the rows. However, it does have integer coefficients if the recurrence of the top row has integer coefficients.

Example: Suppose the top row of the difference triangle is the Pell sequence (see Fig. 2, below).

0	1	2	5	12	29	70	169	408	985	⋯
1	1	3	7	17	41	99	239	577	⋯	
0	2	4	10	24	58	140	338	⋯		
2	2	6	14	34	82	198	⋯			
0	4	8	20	48	116	⋯				
4	4	12	28	68	⋯					
0	8	16	40	⋯						
8	8	24	⋯							
0	16	⋯								
16	⋯									
⋮										

FIGURE 2. Difference triangle for the Pell sequence satisfying $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0, P_1 = 1$. Minimal polynomial is $x^2 - 2x - 1$. Triangle has displacement $(2t, 0)$ with multiplier 2^t for each integer t.

The corresponding recurrence is $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$, so that $c_2 = -1, c_1 = 2, c_0 = 1$. Thus, equation (D″) yields $b_2 = -1, b_1 = 0, b_0 = 2$, and the subsequent equality becomes $d_0^{i+2} - 2d_0^i = 0$. Hence, the recurrence for the left-most column may be written as $d_0^{i+2} = 2d_0^i$. The same recurrence holds for all columns by Theorem 1b or its corollary.

3. POLYNOMIAL OF A SEQUENCE

Definition: We say that $f(x)$ is a *polynomial of a row* (*or column*) *and of the corresponding sequence* $\{a_i\}$ if $f(x) = c_0 + c_1x + \cdots + c_kx^k$ for some nonnegative integer k and some constants $c_0, c_1, \ldots, c_k$, and the equality $c_0a_n + c_1a_{n+1} + \cdots + c_ka_{n+k} = 0$ holds for all $n \geq 0$. Notice that this definition allows $f(x)$ to be the zero polynomial (which is a polynomial of every sequence). Note also that if we express $f(x)$ differently, by adding extra terms with coefficients of zero (thus increasing k), $f(x)$ is still a polynomial of the sequence $\{a_i\}$. If $c_k = 1$, then we say that $f(x)$ is a *characteristic polynomial of the sequence*.

Theorem 3: If $\{d_j^i\}$ is a difference triangle, and $f(x)$ is a polynomial of the top row d^0, then $f(x+1)$ is a polynomial of the left-most column d_0.

Proof: Let $f(x) = c_kx^k + \cdots + c_1x + c_0$. Then, by definition,

$$c_kd_{n+k}^0 + c_{k-1}d_{n+k-1}^0 + \cdots + c_0d_n^0 \quad \text{for } n \geq 0. \tag{E}$$

As in the proof of Theorem 2, we obtain

$$b_kd_0^{m+k} + b_{k-1}d_0^{m+k-1} + \cdots + b_0d_0^m = 0 \quad \text{for } m \geq 0,$$

where the b's are defined by (D″). Then $g(x) = b_kx^k + b_{k-1}x^{k-1} + \cdots + b_1x + b_0$ is a polynomial of d_0.

We may write this in vector notation as

$$g(x) = [b_0, b_1, \ldots, b_k]\begin{bmatrix} 1 \\ x \\ \vdots \\ x^k \end{bmatrix}.$$

Substituting for the b's, using (D″), we obtain

$$g(x) = [c_0, c_1, \ldots, c_{k-1}, c_k]\begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{1} & \cdots & \cdots & \binom{k}{k} \end{bmatrix}\begin{bmatrix} 1 \\ x \\ \vdots \\ x^k \end{bmatrix}$$

$$= [c_0, c_1, \ldots, c_{k-1}, c_k]\begin{bmatrix} 1 \\ 1+x \\ \vdots \\ (1+x)^k \end{bmatrix} = f(x+1),$$

so that $f(x+1)$ is a polynomial of d_0. Clearly, if $f(x)$ has integer coefficients, then so does $f(x+1)$.

Example: Substitution of $x+1$ for x in x^2-2x-1, a characteristic polynomial of the Pell sequence, gives x^2-2 as a characteristic polynomial of the left-most column d_0 of its difference triangle. The corresponding recurrence may be written as $d_0^{i+2}=2d_0^i$, which agrees with the result in the example following Theorem 2.

Lemma 2: Let $f(x)$ and $g(x)$ be polynomials of a sequence $\{a_i\}$, let c be a constant, and let m be a nonnegative integer. Then each of the following is also a polynomial of the sequence $\{a_i\}$:

(a) $f(x)+g(x)$;

(b) $cf(x)$;

(c) $x^m f(x)$.

Proof: These statements follow readily from the definition of a polynomial of a sequence.

Theorem 4: If $f(x)$ is a polynomial of a sequence $\{a_i\}$ and $g(x)$ is any polynomial, then $f(x)g(x)$ is a polynomial of $\{a_i\}$.

Proof: The proof follows from Lemma 2.

Example: The Fibonacci sequence defined by $F_{n+2}=F_{n+1}+F_n$ for $n\geq 0$, $F_0=0, F_1=1$, has x^2-x-1 as a characteristic polynomial. It has $(x^2-x-1)(x+1)=x^3-2x-1$ as another characteristic polynomial corresponding to the recurrence $F_{n+3}=2F_{n+1}+F_n$, which this sequence also satisfied.

Corollary: Let S be a finite set of sequences, each satisfying some LRHCC (not necessarily the same). Then there exists a recurrence that is satisfied by all the sequences.

Proof: Let $f_1(x), f_2(x), \ldots, f_n(x)$ be polynomials of the n sequences in S, respectively. By repeated use of Theorem 4, their product $f_1(x)f_2(x)\cdots f_n(x)$ is a polynomial of each of the sequences in S. A recurrence corresponding to this polynomial is satisfied by every sequence in S.

Example: Let S consist of the Fibonacci and Pell sequences. Characteristic polynomials for the sequences are x^2-x-1 and x^2-2x-1, respectively. Multiplication of these polynomials yields x^4-3x^3+3x+1. The corresponding recurrence is $a_{n+4}=3a_{n+3}-3a_{n+1}-a_n$, which is satisfied by both the Fibonacci and the Pell sequences.

Definition: A characteristic polynomial of a sequence is called a *minimal polynomial* of the sequence if it is of lowest degree. That a minimal polynomial is unique is a consequence of the next theorem.

Theorem 5: A minimal polynomial of a sequence divides every polynomial of the sequence.

Proof: Let $f(x)$ be a minimal polynomial of the sequence $\{a_i\}$ and let $g(x)$ be a polynomial of $\{a_i\}$. Then by the division algorithm, $g(x)=f(x)q(x)+r(x)$, where $q(x)$ and $r(x)$ are polynomials, with $r(x)$ having lower degree than $f(x)$, or else $r(x)\equiv 0$. By Theorem 4, $f(x)q(x)$ is a polynomial of $\{a_i\}$ and, by Lemma 2, so is $r(x) = g(x) - f(x)q(x)$. If $r(x) \not\equiv 0$, then we can multiply $r(x)$ by a constant to get a monic polynomial. By Lemma 2, this is a polynomial of $\{a_i\}$, thereby yielding a characteristic polynomial of degree less than that of $f(x)$, which is a contradiction. Therefore, $r(x) \equiv 0$, and hence $f(x)$ divides $g(x)$. Note that, if $f(x)$ and $g(x)$ have integer coefficients, then so does $q(x)$, since $f(x)$ is monic; thus, divisibility of $g(x)$ by $f(x)$ would also hold in $\mathbf{Z}[x]$.

4. DISPLACEMENTS

Definition: A difference triangle $\{d_j^i\}$ is said to have a *displacement* (s, t) *with multiplier* M if there exist integers s, t and a number M such that the equality

$$d_{n+t}^{m+s} = M d_n^m \tag{F}$$

holds whenever $m, n, m+s,$ and $n+t$ are nonnegative integers. We also say that a sequence has a displacement (s, t) with multiplier M if the difference triangle of which it is the top row has that displacement. If $s = t = 0$, the displacement is called trivial, otherwise nontrivial.

Example: If the Fibonacci sequence is used for the top row, it generates a difference triangle that has displacement (t, t) with multiplier 1 for each integer t (see Fig. 3). The displacements (t, t) may be considered as t multiples of the displacement $(1, 1)$. For an example of a difference triangle whose displacements cannot be expressed as multiples of a single displacement, see Figure 7 at the end of this section.

0	1	1	2	3	5	8	13	21	34	55	⋯
1	0	1	1	2	3	5	8	13	21	⋯	
−1	1	0	1	1	2	3	5	8	⋯		
2	−1	1	0	1	1	2	3	⋯			
−3	2	−1	1	0	1	1	⋯				
5	−3	2	−1	1	0	⋯					
−8	5	−3	2	⋯							
13	−8	5	⋯								
−21	13	⋯									
34	⋯										
⋮											

FIGURE 3. Difference triangle for the Fibonacci sequence satisfying $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0, F_1 = 1$. Minimal polynomial is $x^2 - x - 1$. Triangle has displacement (t, t) with multiplier 1 for each integer t.

Theorem 6: If a difference triangle has a nontrivial displacement, then its top row satisfies some LRHCC.

Proof: Let the difference triangle be $\{d_j^i\}$. First, assume that it has a displacement $(s, t) \neq (0, 0)$ with $s \geq 0$ and with multiplier M. Then we can use the definition of displacement and Lemma 1a to obtain

$$M d_n^0 = d_{n+t}^s = \sum_{\ell=0}^{s} (-1)^\ell \binom{s}{\ell} d_{n+t+s-\ell}^0 \quad \text{for } n \geq \max(0, -t). \tag{G}$$

Subtracting Md_n^0 from the left and right sides of the equation, and then replacing n with $n + \max(0, -t)$, we get an equation of the form (E) in Theorem 3 for some constants $c_k, \ldots, c_1, c_0$, where $k = \max(t+s, s, -t)$. Moreover, the c's are not all 0 except when $s = t = 0$ (and $M = 1$), which is the trivial displacement. Hence, we can write the last equation in the form of some LRHCC which d^0 satisfies.

As a second case, let $s < 0$ and $M \neq 0$. Then we can write (F) in the definition of displacement as

$$d_{n'-t}^{m'-s} = \frac{1}{M} d_{n'}^{m'} \quad \text{for } m', n', m'-s, n'-t \geq 0$$

by substituting $m'-s$ for m and $n'-t$ for n. Thus, $\{d_j^i\}$ also has a displacement $(-s, -t)$ with multiplier $1/M$, and since $-s > 0$, we can use the first case of this proof.

As a final case, suppose that $s < 0$ and $M = 0$. In (F) let $m = -s$ and replace n with $n - t$, to get the equality $d_n^0 = Md_{n-t}^{-s} = 0$, which is valid for all $n \geq \max(0, t)$, so that $d_n^0 = 0$ for each $n \geq \max(0, t)$. Hence, d^0 satisfies any LRHCC of order $\max(0, t)$ or greater with all coefficients zero.

Example: The Pell sequence (Fig. 2) has a displacement $(2, 0)$ with multiplier 2. Hence, (G) becomes

$$2d_n^0 = \sum_{\ell=0}^{2} (-1)^\ell \binom{2}{\ell} d_{n+2-\ell}^0 = d_{n+2}^0 - 2d_{n+1}^0 + d_n^0 \quad \text{for } n \geq 0$$

or

$$d_{n+2}^0 = 2d_{n+1}^0 + d_n^0 \quad \text{for } n \geq 0.$$

Example: The Tribonacci sequence (Fig. 4) has a displacement $(3, 2)$ with multiplier 2. Thus, (G) becomes

$$2d_n^0 = \sum_{\ell=0}^{3} (-1)^\ell \binom{3}{\ell} d_{n+5-\ell}^0 = d_{n+5}^0 - 3d_{n+4}^0 + 3d_{n+3}^0 - d_{n+2}^0 \quad \text{for } n \geq 0$$

or

$$d_{n+5}^0 = 3d_{n+4}^0 - 3d_{n+3}^0 + d_{n+2}^0 + 2d_n^0 \quad \text{for } n \geq 0.$$

In this case, (G) does not give the lowest-order recurrence that the top row satisfies. The corresponding polynomial of d^0, namely, $x^5 - 3x^4 + 3x^3 - x^2 - 2$, is not the minimal polynomial, but has as a factor $x^3 - x^2 - x - 1$, which is the minimal polynomial. The other factor, $x^2 - 2x + 2$, corresponds to the recurrence $d_{n+2}^0 = 2d_{n+1}^0 - 2d_n^0$. Any difference triangle whose top row satisfies the latter will also have a displacement $(3, 2)$ with multiplier 2.

Theorem 7: Let $\{d_j^i\}$ be a difference triangle with displacement (s, t). Let $f(x)$ be the minimal polynomial of d^0. Then, for any two roots α and β of $f(x) = 0$,

$$(\alpha - 1)^s \alpha^t = (\beta - 1)^s \beta^t \tag{H}$$

where 0^0 is defined to be 1.

0	0	1	1	2	4	7	13	24	44	81	149	…
0	1	0	1	2	3	6	11	20	37	68	…	
1	−1	1	1	1	3	5	9	17	31	…		
−2	2	0	0	2	2	4	8	14	…			
4	−2	0	2	0	2	4	6	…				
−6	2	2	−2	2	2	2	…					
8	0	−4	4	0	0	…						
−8	−4	8	−4	0	…							
4	12	−12	4	…								
8	−24	16	…									
−32	40	…										
72	…											
⋮												

FIGURE 4. Difference triangle for a sequence satisfying $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, $T_0 = T_1 = 0$, $T_2 = 1$. Minimal polynomial is $x^3 - x^2 - x - 1$. Triangle has displacement $(3t, 2t)$ with multiplier 2^t for each integer t. Characteristic polynomial is $x^2(x-1)^3 - 2 = x^5 - 3x^4 + 3x^3 - x^2 - 2$ which is divisible by the minimal polynomial $x^3 - x^2 - x - 1$.

Proof: Let M be the multiplier of the displacement. First, consider $s, t \geq 0$. Then (G) can be used to obtain a polynomial $g(x)$ of d^0:

$$g(x) = \sum_{\ell=0}^{s} (-1)^{\ell} \binom{s}{\ell} x^{t+s-\ell} - M = x^t (x-1)^s - M.$$

By Theorem 4, the minimal polynomial $f(x)$ divides $g(x)$, so that any α, β that are zeros of the minimal polynomial $f(x)$ are also zeros of $g(x)$ and, therefore, (H) holds with both sides equal to M. For other cases of s and t, we obtain:

$$g(x) = (x-1)^s - Mx^{-t} \qquad \text{when } s \geq 0, t < 0;$$

$$g(x) = (x-1)^{-s} x^{-t} - \frac{1}{M} \qquad \text{when } s, t < 0, M \neq 0;$$

$$g(x) = (x-1)^{-s} - \frac{1}{M} x^t \qquad \text{when } s < 0, t \geq 0, M \neq 0.$$

It can be verified that all these cases give rise to (H). Note that (H) is always defined because $g(x) = 0$ cannot have a root of 0 when $t < 0$ or a root of 1 when $s < 0$.

If $M = 0$ and $s < 0$, then, as in the last part of the proof of Theorem 6, d^0 contains only a finite number of nonzero terms, and we can derive that $g(x) = x^{\max(0, t)}$. It follows that the minimal polynomial $f(x)$ for d^0 is a power of x and that the roots of $f(x) = 0$ are all 0, so that (H) holds. Theorem 7 is thus true in every case.

Theorem 7 shows that if a sequence with minimal polynomial $f(x)$ has a displacement (s, t), then (H) holds. If $f(x)$ has multiple roots, then the converse need not hold. For example, if $f(x) = (x-2)^3$, then (H) is trivially satisfied for any (s, t), but it can be shown that the corresponding sequence has only the trivial displacement. Theorems 8 and 9 give partial converses of Theorem 7.

Theorem 8: Let $\{d_j^i\}$ be a difference triangle with d^0 satisfying some LRHCC, and let $f(x)$ be the minimal polynomial of d^0. If $f(x)$ divides $g(x)$ of Theorem 7 for some s, t, and M, then $\{d_j^i\}$ has a displacement (s, t) with multiplier M.

Proof: From Theorem 4, $g(x)$ is a polynomial of $\{d^0\}$. Consider the case where $s, t \geq 0$. By the definition of polynomial of a sequence, the left and right sides of (G) are equal. Hence, (F) follows, so that $\{d_j^i\}$ has the displacement claimed. Similar reasoning holds for other cases of s and t.

Theorem 9: Let $\{d_j^i\}$ be a difference triangle with d^0 satisfying some LRHCC, and let $f(x)$ be the minimal polynomial of d^0. Suppose that $f(x)=0$ has no multiple roots. Then $\{d_j^i\}$ has a displacement (s, t) if (H) holds for every pair of roots α and β of $f(x)=0$.

Proof: By (H), $(\alpha-1)^s\alpha^t$ has the same value for any α that is a root of $f(x)=0$. Call this value M and substitute it in $g(x)$ as defined in Theorem 7. Every root of $f(x)=0$ is a root of $g(x)=0$. Since $f(x)=0$ has no multiple roots, $f(x)$ must divide $g(x)$, so that the result follows by an application of Theorem 8.

Theorems 7-9 can be useful in determining what displacements a sequence has. Some examples follow.

Example A: A sequence satisfies $a_{n+2}=a_{n+1}+ca_n$, $c \geq 0$, $a_0=0, a_1 \neq 0$. The minimal polynomial, x^2-x-c, has two zeros:

$$\alpha=\frac{1+\sqrt{4c+1}}{2} \quad \text{and} \quad \beta=\frac{1-\sqrt{4c+1}}{2}.$$

Substitution of these values into (H) leads to

$$\left(\frac{1+\sqrt{4c+1}}{2}\right)^t\left(\frac{-1+\sqrt{4c+1}}{2}\right)^s=\left(\frac{1-\sqrt{4c+1}}{2}\right)^t\left(\frac{-1-\sqrt{4c+1}}{2}\right)^s$$

to be solved for integers s and t. Multiplying both sides by $2^{s+t}(-1)^s$, we obtain

$$(1+\sqrt{4c+1})^t(1-\sqrt{4c+1})^s=(1-\sqrt{4c+1})^t(1+\sqrt{4c+1})^s.$$

The only solutions to this equation occur when $s=t$. Thus, the only nontrivial displacements for a sequence satisfying the given recurrence conditions are (t, t); e.g., $(1, 1)$ is a displacement for the Fibonacci sequence when $c=1$ and $a_1=1$.

Example B: A sequence satisfies $a_{n+2}=ca_{n+1}+a_n$, where $c>2$, $a_0=0, a_1 \neq 0$. The minimal polynomial x^2-cx-1 has zeros $\frac{c}{2} \pm \frac{\sqrt{c^2+4}}{2}$, so that (H) becomes

$$\left(\frac{c+\sqrt{c^2+4}}{2}\right)^t\left(\frac{c-2+\sqrt{c^2+4}}{2}\right)^s=\left(\frac{c-\sqrt{c^2+4}}{2}\right)^t\left(\frac{c-2-\sqrt{c^2+4}}{2}\right)^s.$$

Multiplying both sides by 2^{s+t}, and rearranging terms, we get

$$\left(\frac{c+\sqrt{c^2+4}}{c-\sqrt{c^2+4}}\right)^t = \left(\frac{c-2-\sqrt{c^2+4}}{c-2+\sqrt{c^2+4}}\right)^s.$$

Since $c>2$, $c-2>0$, and $c^2+4>0$, we find that

$$\left|\frac{c+\sqrt{c^2+4}}{c-\sqrt{c^2+4}}\right|>1 \quad \text{and} \quad \left|\frac{c-2-\sqrt{c^2+4}}{c-2+\sqrt{c^2+4}}\right|<1.$$

Given these inequalities, the only way that the above equality can hold if $s, t \geq 0$ (or $s, t \leq 0$) is that $s=t=0$. Hence, a sequence satisfying such a recurrence has only the nontrivial displacement when s and t are both nonnegative or both nonpositive.

Example C: The "Mersenne sequence" (Robbins [4], p. 194), given by the formula $M_n = 2^n - 1$, satisfies the recurrence $M_{n+2} = 3M_{n+1} - 2M_n$, $M_0 = 0, M_1 = 1$. The minimal polynomial, $x^2 - 3x + 2$, has $\alpha = 1$ and $\beta = 2$ as zeros. Using these values in (H), we obtain

$$(0)^s(1)^t = (1)^s(2)^t,$$

which cannot hold if $s \neq 0$, since that would yield $0 = 2^t$. If $s = 0$, then $1 = 2^t$, which indicates that $t = 0$ as well. This shows that the "Mersenne sequence" has only the trivial displacement.

Further examples of difference triangles and their displacements are considered in Figures 5, 6, and 7. Figure 5 shows a difference triangle for a sequence satisfying $a_i = -na_{i-1}$ with $a_0 = 1$. The minimal polynomial is $x + n = 0$. The difference triangle has displacement (s, t) with multiplier $(-1)^{s+t}(n+1)^s n^t$ for all integers s and t. Figure 6 shows a difference triangle generated by a sequence $\{a_i\}$ satisfying $a_{n+2} = 4a_{n+1} - 2a_n$ with $a_0 = 0$ and $a_1 = 1$. The minimal polynomial is $x^2 - 4x + 2$. The difference triangle has displacement $(2t, -2t)$ with multiplier 2^t for each integer t. Figure 7 shows a difference triangle generated by a sequence $\{a_i\}$ satisfying $a_{n+2} = 2a_{n+1} - 2a_n$ with $a_0 = 0$ and $a_1 = 1$. The minimal polynomial is $x^2 - 2x + 2$. Two displacements are $(0, 4)$ with multiplier -4 and $(1, 2)$ with multiplier -2. They are *independent* in the sense that a difference triangle has independent displacements (s, t) and (s', t') if $st' \neq ts'$. The authors are investigating conditions under which a difference triangle has independent displacements.

$$\begin{array}{cccccc}
1 & -n & n^2 & -n^3 & n^4 & -n^5 \\
-(n+1) & n(n+1) & -n^2(n+1) & n^3(n+1) & -n^4(n+1) & \cdots \\
(n+1)^2 & -n(n+1)^2 & n^2(n+1)^2 & -n^3(n+1)^2 & \cdots & \\
-(n+1)^3 & n(n+1)^3 & -n^2(n+1)^3 & \cdots & & \\
(n+1)^4 & -n(n+1)^4 & \cdots & & & \\
-(n+1)^5 & \cdots & & & & \\
\vdots & & & & &
\end{array}$$

FIGURE 5. Difference triangle for a sequence satisfying $a_i = -na_{i-1}$ with $a_0 = 1$. Minimal polynomial is $x + n$. Triangle has displacement (s, t) with multiplier $(-1)^{s+t}(n+1)^s n^t$ for all integers s and t.

0	1	4	14	48	164	560	⋯
1	3	10	34	116	396	⋯	
2	7	24	82	280	⋯		
5	17	58	198	⋯			
12	41	140	⋯				
29	99	⋯					
70	⋯						
⋮							

FIGURE 6. Difference triangle generated by the sequence $\{a_i\}$ satisfying $a_{n+2} = 4a_{n+1} - 2a_n$ with $a_0 = 0$ and $a_1 = 1$. Minimal polynomial is $x^2 - 4x + 2$. Triangle has displacement $(2t, -2t)$ with multiplier 2^t for each integer t.

0	1	2	2	0	−4	−8	−8	0	16	⋯
1	1	0	−2	−4	−4	0	8	16	⋯	
0	−1	−2	−2	0	4	8	8	⋯		
−1	−1	0	2	4	4	0	⋯			
0	1	2	2	0	−4	⋯				
1	1	0	−2	−4	⋯					
0	−1	−2	−2	⋯						
−1	−1	0	⋯							
0	1	⋯								
1	⋯									
⋮										

FIGURE 7. Difference triangle generated by the sequence $\{a_i\}$ satisfying $a_{n+2} = 2a_{n+1} - 2a_n$ with $a_0 = 0$ and $a_1 = 1$. Minimal polynomial is $x^2 - 2x + 2$. Two independent displacements are (0, 4) with multiplier −4 and (1, 2) with multiplier −2.

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