ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-802 ***Proposed by Al Dorp, Edgemere, NY***

For $n > 0$, let $T_n = n(n+1)/2$ denote the n^{th} triangular number. Find a formula for T_{2n} in terms of T_n.

B-803 ***Proposed by Herta T. Freitag, Roanoke, VA***

For n even and positive, evaluate

$$\sum_{i=0}^{n/2} \binom{n}{i} L_{n-2i}.$$

B-804 ***Proposed by the editor***

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d}.$$

B-805 ***Proposed by David Zeitlin, Minneapolis, MN***

Solve the recurrence $P_{n+6} = P_{n+5} + P_{n+4} - P_{n+2} + P_{n+1} + P_n$, for $n \geq 0$, with initial conditions $P_0 = 1$, $P_1 = 1$, $P_2 = 2$, $P_3 = 3$, $P_4 = 4$, and $P_5 = 7$.

B-806 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

(a) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x+x^3}$ is the difference of two Fibonacci numbers.

(b) Show that the coefficient of every term in the expansion of $\frac{x}{1-2x-2x^2+x^3}$ is the product of two consecutive Fibonacci numbers.

B-807 ***Proposed by R. André-Jeannin, Longwy, France***

The sequence $\langle W_n \rangle$ is defined by the recurrence $W_n = PW_{n-1} - QW_{n-2}$, for $n \ge 2$, with initial conditions $W_0 = a$ and $W_1 = b$, where a and b are integers and P and Q are odd integers. Prove that, for $k \ge 0$,

$$W_{n+3\cdot 2^k} \equiv W_n \pmod{2^{k+1}}.$$

SOLUTIONS

An Integral Ratio

B-772 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Prove that

$$\frac{L_n^2 + L_{n+a}^2}{F_n^2 + F_{n+a}^2}$$

is always an integer if a is odd. How should this problem be modified if a is even?

Solution by C. Georghiou, University of Patras, Patras, Greece

Identity (I_{12}) of [1] reads $5F_n^2 = L_n^2 - 4(-1)^n$. It follows that, for a odd,

$$L_n^2 + L_{n+a}^2 = 5(F_n^2 + F_{n+a}^2);$$

whereas, for a even,

$$L_n^2 - L_{n+a}^2 = 5(F_n^2 - F_{n+a}^2);$$

and the integer, in both cases, is 5. In other words, for $a \ne 0$, we have

$$\frac{L_n^2 - (-1)^a L_{n+a}^2}{F_n^2 - (-1)^a F_{n+a}^2} = 5.$$

When a is even, Zeitlin and Filipponi (independently) found the formula

$$\frac{L_n^2 + L_{n+a}^2 - 8(-1)^n}{F_n^2 + F_{n+a}^2} = 5.$$

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Michel A. Ballieu, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David C. Terr, David Zeitlin, and the proposer.

Zecky Would Be Proud

B-773 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 32, no. 5, November 1994)

Find the number of terms in the Zeckendorf representation of $S_n = \sum_{i=1}^n F_i^2$.
[The Zeckendorf representation of an integer expresses that integer as a sum of distinct nonconsecutive Fibonacci numbers.]

Solution by David C. Terr, University of California, Berkeley, CA

We claim that

$$S_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} F_{2n-4k-1}$$

is the Zeckendorf representation of S_n. It consists of $\lfloor (n+1)/2 \rfloor$ terms.

Proof: Clearly this equation holds for $n=1$ and $n=2$. Thus, it is enough to show that $S_n - S_{n-2} = F_{2n-1}$ for $n>1$. But $S_n - S_{n-2} = F_n^2 + F_{n-1}^2$ and $F_n^2 + F_{n-1}^2 = 5F_{2n-1}$ by identity (I_{11}) of [1]. Thus, the result follows.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

The proposer also found that the number of terms in the Zeckendorf representation of $\sum_{i=1}^{n} L_i^2$ is n. A very nice result. No other reader came up with any generalization of this problem.

Also solved by Michel A. Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, David Zeitlin, and the proposer.

A Congruence for the Period

B-774 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Let $\langle H_n \rangle$ be any sequence of integers such that $H_{n+2} = H_{n+1} + H_n$ for all n. Let p and m be positive integers such that $H_{n+p} \equiv H_n \pmod{m}$ for all integers n. Prove that the sum of any p consecutive terms of the sequence is divisible by m.

Solution by Leonard A. G. Dresel, Reading, England

Let $S(a)$ be the sum of p consecutive terms of the sequence starting with H_a. Using the recurrence relation $H_n = H_{n+2} - H_{n+1}$, we have

$$\begin{aligned} S(a) &= H_a + H_{a+1} + \cdots + H_{a+p-1} \\ &= (H_{a+2} - H_{a+1}) + (H_{a+3} - H_{a+2}) + \cdots + (H_{a+p+1} - H_{a+p}) \\ &= H_{a+p+1} - H_{a+1} \quad \text{(since all other terms cancel)} \\ &\equiv 0 \pmod{m}, \end{aligned}$$

for any value of a. Therefore, the sum of any p consecutive terms is divisible by m.

Also solved by Michel Ballieu, Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David Zeitlin, and the proposer.

Golden Powers

B-775 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Let $g = \alpha + 2$. Express g^{17} in the form $p\alpha + q$, where p and q are integers.

Solution by Norbert Jensen, Kiel, Germany

We show that

$$g^{2n} = 5^n(F_{2n}\alpha + F_{2n-1})$$
$$g^{2n+1} = 5^n(L_{2n+1}\alpha + L_{2n})$$

for all integers n. It then follows that $g^{17} = 5^8 L_{17}\alpha + 5^8 L_{16} = 1394921875\alpha + 862109375$.

Proof of the general assertion. We use the following identities:

(i) $$g = \alpha + 2 = \alpha^2 + 1 = \alpha(\alpha - \beta) = \alpha\sqrt{5}.$$

(ii) $$\alpha^n = F_n\alpha + F_{n-1}.$$

(iii) $$F_n + F_{n+2} = L_{n+1}$$

Both (ii) and (iii) are easily proved by induction on n (or they can be found in [1] on pages 52 and 24, respectively).

Applying (i) and (ii), we obtain

$$g^{2n} = 5^n\alpha^{2n} = 5^n(F_{2n}\alpha + F_{2n-1}).$$

Therefore, by (iii), $g^{2n+1} = 5^n(F_{2n}\alpha + F_{2n-1})(\alpha + 2) = 5^n[(3F_{2n} + F_{2n-1})\alpha + F_{2n} + 2F_{2n-1}] = 5^n[(F_{2n} + F_{2n+2})\alpha + F_{2n-1} + F_{2n+1}] = 5^n(L_{2n+1}\alpha + L_{2n})$.

Reference

1. S. Vajda. Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications. Chichester, England: Ellis Horwood Ltd., 1989.

Generalization 1 by David Zeitlin, Minneapolis, MN

Let a and b be distinct roots of $x^2 = Px - Q$. Let $\langle U_n\rangle$ and $\langle V_n\rangle$ be sequences defined by the recurrences $U_{n+2} = PU_{n+1} - QU_n, U_0 = 0, U_1 = 1$, and $V_{n+2} = PV_{n+1} - QV_n, V_0 = 2, V_1 = P$. We will show that

$$(Pa - 2Q)^{2n+1} = (P^2 - 4Q)^n(aV_{2n+1} - QV_{2n})$$

for all nonnegative integers n. (In our problem, $P = 1, Q = -1, a = \alpha, V_n = L_n$, and $n = 8$.)

To prove this identity, we note that, since $Q = ab, Pa - 2Q = (Pa - Q) - Q = a^2 - ab = a(a - b) = a(P^2 - 4Q)^{1/2}$. Thus, $(Pa - 2Q)^{2n+1} = a^{2n+1}(P^2 - 4Q)^{n+1/2}$. Since $a^k = aU_k - QU_{k-1}$,

$$\begin{aligned}(P^2 - 4Q)^{1/2}a^{2n+1} &= (P^2 - 4Q)^{1/2}(aU_{2n+1} - QU_{2n})\\ &= a(a^{2n+1} - b^{2n+1}) - Q(a^{2n} - b^{2n})\\ &= a^{2n+2} - Qb^{2n} - Qa^{2n} + Qb^{2n} = a^{2n+2} - Qa^{2n}\\ &= (aU_{2n+2} - QU_{2n+1}) - Q(aU_{2n} - QU_{2n-1})\\ &= a(U_{2n+2} - QU_{2n}) - Q(U_{2n+1} - QU_{2n-1})\\ &= aV_{2n+1} - QV_{2n}, \quad \text{since } V_n = U_{n+1} - QU_{n-1}.\end{aligned}$$

This proves the result. In the same manner, we can also prove

$$(Pa - 2Q)^{2n} = (P^2 - 4Q)^n(aU_{2n} - QU_{2n-1}),$$
$$(Pb - 2Q)^{2n+1} = (P^2 - 4Q)^n(bV_{2n+1} - QV_{2n}),$$
$$(P_b - 2Q)^{2n} = (P^2 - 4Q)^n(bU_{2n} - QU_{2n-1}).$$

Generalization 2 by Murray S. Klamkin, University of Alberta, Canada

We will show that, if m and r are given integers (with $r>0$) and if we let $g=\alpha+m$, then g^r can be written as $p\alpha+q$ for some integers p and q. Let $g^r=p_r\alpha+q_r$. Then

$$\begin{aligned}g^{r+1}&=(p_r\alpha+q_r)(\alpha+m)=p_{r+1}\alpha+q_{r+1}\\&=p_r\alpha^2+(mp_r+q_r)\alpha+mq_r=p_r(\alpha+1)+(mp_r+q_r)\alpha+mq_r.\end{aligned}$$

Hence, $p_{r+1}=(m+1)p_r+q_r$ and $q_{r+1}=p_r+mq_r$. It then follows that

$$p_{r+2}=(2m+1)p_{r+1}-(m^2+m-1)p_r \quad\text{and}\quad q_{r+2}=(2m+1)q_{r+1}-(m^2+m-1)q_r$$

with $p_0=0, q_0=1, p_1=1,$ and $q_1=m$. Thus, p_r and q_r are integers for all positive integers r. *Klamkin went on to give explicit formulas for p_r and q_r. Anderson found the same generalization as Klamkin.*

Also solved by Mark Anderson, Michel Ballieu, Glenn A. Bookout, Scott H. Brown, Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Carl Libis, Bob Prielipp, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David C. Terr, and the proposer.

An Even Sum

B-776 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Find all values of n for which $S_n=\sum_{k=1}^{n}kF_k$ is even.

Solution by Paul S. Bruckman, Edmonds, WA

The periodic sequences $\langle n \pmod 2\rangle$ and $\langle F_n \pmod 2\rangle$ for $n>0$ have periods 2 and 3, respectively. Therefore, the sequence $\langle nF_n \pmod 2\rangle$ has period 6, as does the sequence $\langle S_n \pmod 2\rangle$. We may then form the following table:

n	F_n	$nF_n \pmod 2$	$S_n \pmod 2$
1	1	1	1
2	1	0	1
3	2	0	1
4	3	0	1
5	5	1	0
6	8	0	0

By the foregoing comments, and by inspection of the table above, we conclude that S_n is even if and only if n is congruent to 0 or 5 modulo 6.

Also solved by Charles Ashbacher, Michel Ballieu, Charles K. Cook, Leonard A. G. Dresel, Piero Filipponi, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Carl Libis, H.-J. Seiffert, Sahib Singh, Lawrence Somer, M. N. S. Swamy, David Zeitlin, and the proposer.

A Tricky Congruence Criterion

B-777 ***Proposed by Herta Freitag, Roanoke, VA (Vol. 32, no. 5, November 1994)***

Find all integers a such that $n\equiv a \pmod 4$ if and only if $L_n\equiv a \pmod 5$.

Solution by Paul S. Bruckman, Edmonds, WA

As is easily verified, the periodic sequence $\langle L_n \pmod 5\rangle_{n\geq 0} = (2,1,3,4,2,1,3,4,\ldots)$ has period equal to 4. The solutions of our problem are therefore those values of a modulo 20 such that any of the following conditions holds:

(i) $a \equiv 0 \pmod 4$ and $a \equiv L_0 = 2 \pmod 5$;
(ii) $a \equiv 1 \pmod 4$ and $a \equiv L_1 = 1 \pmod 5$;
(iii) $a \equiv 2 \pmod 4$ and $a \equiv L_2 = 3 \pmod 5$;
(iv) $a \equiv 3 \pmod 4$ and $a \equiv L_3 = 4 \pmod 5$.

The solutions of (i)-(iv), respectively, are as follows: $a \equiv 12, 1, 18, \text{ or } 19 \pmod{20}$. Therefore, $n \equiv a \pmod 4$ if and only if $L_n \equiv a \pmod 5$, whenever $a \equiv 1, 12, 18, \text{ or } 19 \pmod{20}$, and for no other values of a.

A related fact: $n \equiv a \pmod 4$ if and only if $L_n \equiv L_a \pmod 5$, appeared in this Quarterly **32.3** (1994):245.

Also solved by Charles Ashbacher, Norbert Jensen, H.-J. Seiffert, Lawrence Somer, David C. Terr, and the proposer. Six incorrect solutions were received.

Fibonacci's Last Theorem

B-778 ***Proposed by Eliot Jacobson, Ohio University, Athens, OH (Vol. 33, no. 1, February 1995)***

Show that the equation $x^n + y^n = z^n$ has no nontrivial solutions consisting entirely of Fibonacci numbers, for $n \geq 2$.

Solution by the proposer

Let $z = F_c$, with $c \geq 3$. Using the fact that $F_cF_{c-2} = F_{c-1}^2 + (-1)^{c-1}$, which is Identity ($I_{13}$) from [1], observe that

$$\begin{aligned} F_c^2 &= F_c(F_{c-1} + F_{c-2}) \\ &= F_cF_{c-1} + F_cF_{c-2} \\ &\geq (1 + F_{c-1})F_{c-1} + F_cF_{c-2} \\ &= F_{c-1} + F_{c-1}^2 + F_{c-1}^2 + (-1)^{c-1} \\ &> F_{c-1}^2 + F_{c-1}^2. \end{aligned}$$

Since $F_c > F_{c-1}$, it follows that $F_c^n > F_{c-1}^n + F_{c-1}^n$, so the equality can never hold.

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Several solvers noted that it suffices to consider positive Fibonacci numbers in light of the identity $F_{-k} = (-1)^{k+1}F_k$. Seiffert also proved Lucas's Last Theorem: If $n \geq 2$ is an integer, then the equation $x^n + y^n = z^n$ has no solution in Lucas numbers, x, y, and z. Prielipp noted that this problem is equivalent to Theorem 6 in L. Carlitz, "A Note on Fibonacci Numbers," this Quarterly **2.1** *(1964):15-28.*

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, Bob Prielipp, and H.-J. Seiffert. Andrew Wiles could not be reached for comment.

Find the Identity

B-779 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 33, no. 1, February 1995)

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 6F_{n-b} + F_{n-c} + F_{n-d}.$$

Editorial Comment: Most solvers pulled the answer out of a hat:

$$F_n = F_{n-2} + 6F_{n-5} + F_{n-6} + F_{n-8}.$$

The proof by induction is then straightforward. Only Bruckman and Georghióu submitted a proof that showed how to find a, b, c, and d; but their methods do not seem to generalize. So to test your prestidigitation abilities, the editor has concocted a related problem (see Problem B-804 in this issue). Let's see who can pull a rabbit out of *that* hat.

Solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, David Zeitlin, and the proposer.

Production Inequality

B-780 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***
(Vol. 33, no. 1, February 1995)

Prove that:

(a) $$F_1 \cdot F_2 \cdot F_3 \cdot \cdots \cdot F_n \le \exp(F_{n+2} - n - 1);$$
(b) $$F_1 \cdot F_3 \cdot F_5 \cdot \cdots \cdot F_{2n-1} \le \exp(F_{2n} - n);$$
(c) $$F_2 \cdot F_4 \cdot F_6 \cdot \cdots \cdot F_{2n} \le \exp(F_{2n+1} - n - 1).$$

Solution by David Zeitlin, Minneapolis, MN

All proofs are by mathematical induction. Since $1 + x \le \exp(x)$ for $x > 0$, we get (letting $y = x + 1$): $y \le \exp(y - 1)$ for $y > 1$. We will use this inequality repeatedly, below. The results are easily seen to be true for $n = 1$ and $n = 2$, so we need only give the induction step.

Proof of (a):
$$\begin{aligned}(F_1F_2F_3 \ldots F_n)F_{n+1} &\le F_{n+1} \cdot \exp(F_{n+2} - n - 1)\\ &\le \exp(F_{n+1} - 1) \cdot \exp(F_{n+2} - n - 1)\\ &= \exp(F_{n+1} - 1 + F_{n+2} - n - 1) = \exp(F_{n+3} - (n+1) - 1).\end{aligned}$$

Proof of (b):
$$\begin{aligned}(F_1F_3F_5 \ldots F_{2n-1})F_{2n+1} &\le F_{2n+1} \cdot \exp(F_{2n} - n)\\ &\le \exp(F_{2n+1} - 1) \cdot \exp(F_{2n} - n)\\ &= \exp(F_{2n+1} - 1 + F_{2n} - n) = \exp(F_{2(n+1)} - (n+1)).\end{aligned}$$

Proof of (c):
$$\begin{aligned}(F_2F_4F_6 \ldots F_{2n})F_{2n+2} &\le F_{2n+2} \cdot \exp(F_{2n+1} - n - 1)\\ &\le \exp(F_{2n+2} - 1) \cdot \exp(F_{2n+1} - n - 1)\\ &= \exp(F_{2n+2} - 1 + F_{2n+1} - n - 1) = \exp(F_{2(n+1)+1} - (n+1) - 1).\end{aligned}$$

Generalization 1 by H.-J. Seiffert, Berlin, Germany

We shall prove that for all positive integers n,

(a′) $$F_1 \cdot F_2 \cdot F_3 \cdot \cdots \cdot F_n \le 2^{F_{n+2} - n - 1};$$
(b′) $$F_1 \cdot F_3 \cdot F_5 \cdot \cdots \cdot F_{2n-1} \le 2^{F_{2n} - n};$$
(c′) $$F_2 \cdot F_4 \cdot F_6 \cdot \cdots \cdot F_{2n} \le 3^{(F_{2n+1} - n - 1)/2}.$$

Since $F_{2n+1} > F_{2n} \geq n$, $F_{n+2} \geq n+1$, $n \geq 1$; and $\sqrt{3} < 2 < e$, it is obvious that these inequalities imply the proposed ones. In (a′) and (b′), the base 2 cannot be replaced by a smaller base, as is readily seen by setting $n = 3$ and $n = 2$, respectively. Similarly, in (c′), the base $\sqrt{3}$ is best possible, as one can see by taking $n = 2$.

From identities (I_1), (I_5), and (I_6) of [1], we obtain

$$\sum_{k=1}^{n}(F_k - 1) = F_{n+2} - n - 1, \tag{1}$$

$$\sum_{k=1}^{n}(F_{2k-1} - 1) = F_{2n} - n, \tag{2}$$

and

$$\sum_{k=1}^{n}(F_{2k} - 1) = F_{2n+1} - n - 1. \tag{3}$$

It is easily seen that $m \leq 2^{m-1}$ for all positive integers m. Thus, (a′) follows by considering the product of the inequalities $F_k \leq 2^{F_k - 1}$, $k = 1, 2, 3, \ldots, n$, and applying (1). Similarly, (b′) follows from the inequalities $F_{2k-1} \leq 2^{F_{2k-1}-1}$, $k = 1, 2, 3, \ldots, n$, and (2). For the proof of (c′), we note that $m \leq 3^{(m-1)/2}$ for all positive integers m such that $m \neq 2$. Since no Fibonacci number with even subscript equals 2, (c′) follows from $F_{2k} \leq 3^{(F_{2k}-1)/2}$, $k = 1, 2, 3, \ldots, n$, and (3).

Generalization 2 by H.-J. Seiffert, Berlin, Germany

We will show that

$$\prod_{k=1}^{n} F_{2k} \leq \begin{cases} F_{n+1}^{n}, & \text{for odd } n, \\ (L_{n+1}/\sqrt{5})^n, & \text{for even } n. \end{cases}$$

[Also: lower bounds are $(2/e)^n$ times the upper bounds.] This follows by taking $q = \beta^2/\alpha^2$ in the inequality

$$(2/e)^n(1 - q^{(n+1)/2})^n \leq \prod_{k=1}^{n}(1 - q^k) \leq (1 - q^{(n+1)/2})^n,$$

which holds for all $q \in (0, 1)$ and all positive integers n. This inequality comes from [2].

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.
2. H.-J. Seiffert. "Problem 4406." *School Science and Mathematics* **94.1** (1994):54.

Also solved by Šefket Arslahagić, Paul S. Bruckman, Charles K. Cook, L. A. G. Dresel, C. Georghiou, Pentti Haukkanen, Russell J. Hendel, Norbert Jensen, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, H.-J. Seiffert, and the proposer.

Addendum: Adam Stinchcombe was inadvertently omitted as a solver of Problems B-769 and B-771.

❖❖❖