JACOBSTHAL REPRESENTATION NUMBERS

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1. INTRODUCTION

Two sequences of numbers concern us, namely, the *Jacobsthal sequence* $\{J_n\}$ (see [4]) defined by

$$J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0, J_1 = 1, \ n \geq 0, \tag{1.1}$$

and the *Jacobsthal-Lucas sequence* $\{j_n\}$ defined by

$$j_{n+2} = j_{n+1} + 2j_n, \quad j_0 = 2, j_1 = 1, \ n \geq 0. \tag{1.2}$$

Applications of these two sequences to curves are given in [4]. Sequence (1.1) appears in [11], but (1.2) does not.

From (1.1) and (1.2) we thus have the following tabulation for the *Jacobsthal numbers* J_n and the *Jacobsthal-Lucas numbers* j_n:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
J_n	0	1	1	3	5	11	21	43	85	171	341	$\cdots$
j_n	2	1	5	7	17	31	65	127	257	511	1025	$\cdots$

(1.3)

When required, we can extend these sequences through negative values of n by means of the recurrences (1.1) and (1.2). Observe that all the J_n and j_n (except j_0) are odd, by virtue of the definitions.

Recurrences (1.1) and (1.2) involve the characteristic equation

$$x^2 - x - 2 = 0 \tag{1.4}$$

with roots

$$\alpha = 2, \quad \beta = -1 \tag{1.5}$$

so that

$$\alpha + \beta = 1, \quad \alpha\beta = -2, \quad \alpha - \beta = 3. \tag{1.6}$$

Wherever it is sensible to do so, we will replace α, β by $2, -1$, respectively.

Explicit closed form expressions for J_n and j_n are $(n \geq 1)$

$$J_n = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-r}{r} 2^r \tag{1.7}$$

(see [3]) and

$$j_n = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-r} \binom{n-r}{r} 2^r. \tag{1.8}$$

Induction on n provides the required proofs.

In the theory of minimal and maximal representations of nonnegative integers by elements of a sequence $\{a_n\}$ (e.g., Fibonacci or Pell numbers; see [2], [7], [8]), we discover the importance of a new sequence whose members are those integers that can be represented both minimally and maximally by a sum of elements of $\{a_n\}$ for which the coefficients are all unity.

It is the object of this article to investigate the corresponding new sequences associated with $\{J_n\}$ and $\{j_n\}$.

But first we establish a few basic properties of the sequences (1.1) and (1.2), some of which will find subsequent application as our theme develops.

2. BASIC PROPERTIES OF THE JACOBSTHAL NUMBERS

Initially, these properties enter our mathematical Noah's Ark in pairs, as did (1.7) and (1.8). Standard techniques may be used to generate them and their numerous progeny, the most handsome of which is (2.9).

Generating functions

$$\sum_{i=1}^{\infty} J_i x^{i-1} = (1-x-2x^2)^{-1} \quad \text{(cf. [3])}, \tag{2.1}$$

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1+4x)(1-x-2x^2)^{-1}. \tag{2.2}$$

Binet forms

$$J_n = \frac{\alpha^n - \beta^n}{3} = \frac{1}{3}(2^n - (-1)^n), \tag{2.3}$$

$$j_n = \alpha^n + \beta^n = 2^n + (-1)^n. \tag{2.4}$$

Simson formulas

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1}, \tag{2.5}$$

$$j_{n+1}j_{n-1} - j_n^2 = 9(-1)^{n-1}2^{n-1} = -9(J_{n+1}J_{n-1} - J_n^2). \tag{2.6}$$

Summation formulas

$$\sum_{i=2}^{n} J_i = \frac{J_{n+2}-3}{2}, \tag{2.7}$$

$$\sum_{i=1}^{n} j_i = \frac{j_{n+2}-5}{2}. \tag{2.8}$$

The significance of the lower bound for i in the useful formulas (2.7) and (2.8) will become apparent later in Sections 4 and 5.

Interrelationships

$$j_n J_n = J_{2n}, \tag{2.9}$$

$$j_n = J_{n+1} + 2J_{n-1}, \tag{2.10}$$

$$9J_n = j_{n+1} + 2j_{n-1}, \tag{2.11}$$

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3.2^n, \tag{2.12}$$

$$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1}, \tag{2.13}$$

$$j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) = 3(-1)^{n+1}. \tag{2.14}$$

$$2j_{n+1} + j_{n-1} = 3(2J_{n+1} + J_{n-1}) + 6(-1)^{n+1}, \tag{2.15}$$

$$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} = 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r}, \tag{2.16}$$

$$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1), \tag{2.17}$$

$$j_n = 3J_n + 2(-1)^n \quad \text{[cf. (2.12)]}, \tag{2.18}$$

$$3J_n + j_n = 2^{n+1}, \tag{2.19}$$

$$J_n + j_n = 2J_{n+1}, \tag{2.20}$$

$$\lim_{n\to\infty}\left(\frac{J_{n+1}}{J_n}\right) = \lim_{n\to\infty}\left(\frac{j_{n+1}}{j_n}\right) = 2, \tag{2.21}$$

$$\lim_{n\to\infty}\left(\frac{j_n}{J_n}\right) = 3, \tag{2.22}$$

$$j_{n+2}j_{n-2} - j_n^2 = -9(J_{n+2}J_{n-2} - J_n^2) = 9(-1)^n 2^{n-2}, \tag{2.23}$$

$$J_m j_n + J_n j_m = 2J_{m+n} \quad [m = n \to (2.9)], \tag{2.24}$$

$$j_m j_n + 9J_m J_n = 2j_{m+n}, \tag{2.25}$$

$$j_n^2 + 9J_n^2 = 2j_{2n} \quad [m = n \text{ in } (2.25)], \tag{2.26}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n}, \tag{2.27}$$

$$j_m j_n - 9J_m J_n = (-1)^n 2^{n+1} j_{m-n}, \tag{2.28}$$

$$j_n^2 - 9J_n^2 = (-1)^n 2^{n+2} \quad [m = n \text{ in } (2.28)]. \tag{2.29}$$

Economies of space (and cost!) preclude the addition of further properties which may be of lesser interest and value. Observe, however, that (2.9) is an important feature of $\{J_n\}$ and $\{j_n\}$, being analogous to $F_nL_n = F_{2n}$ and $P_nQ_n = P_{2n}$ for Fibonacci and Lucas numbers, and Pell and Pell-Lucas numbers, respectively. One might remark, in passing that the infinite limit of our $\frac{1}{2}Q_n / P_n$ [cf. (2.22)] is mentioned in [12] in dealing with irrationality.

Associated Sequences

Invoking [6], we define the k^{th} *associated sequences* $\{J_n^{(k)}\}$ and $\{j_n^{(k)}\}$ of $\{J_n\}$ and $\{j_n\}$ to be, respectively, given by $(k \geq 1)$

$$J_n^{(k)} = J_{n+1}^{(k-1)} + 2J_{n-1}^{(k-1)} \tag{2.30}$$

and

$$j_n^{(k)} = j_{n+1}^{(k-1)} + 2j_{n-1}^{(k-1)}, \tag{2.31}$$

where $J_n^{(0)} = J_n$, $j_n^{(0)} = j_n$. Accordingly,

$$J_n^{(1)} = j_n \quad \text{by (2.10)} \tag{2.32}$$

and

$$j_n^{(1)} = 9J_n \quad \text{by (2.11)} \tag{2.33}$$

are the generic members of the *first associated sequences* $\{J_n^{(1)}\}$ and $\{j_n^{(1)}\}$.

Deducing the following neat results is an easy matter on appeal to (2.10) and (2.11):

$$J_n^{(2m)} = 3^{2m} J_n, \tag{2.34}$$

$$J_n^{(2m+1)} = 3^{2m} j_n, \tag{2.35}$$

$$j_n^{(2m)} = 3^{2m} j_n, \tag{2.36}$$

$$j_n^{(2m-1)} = 3^{2m} J_n. \tag{2.37}$$

Expressed succinctly,

$$\left.\begin{array}{l} J_n^{(2m)} = j_n^{(2m-1)} \\ j_n^{(2m)} = J_n^{(2m+1)} \end{array}\right\}. \tag{2.38}$$

Analogous results to (2.34)-(2.37) for Fibonacci and Lucas numbers are stated in [6]. Pairs of results like these can be incorporated into a more general system for polynomials that extends to negative values of m and n. Material on this research has been submitted for publication.

3. JACOBSTHAL REPRESENTATION SEQUENCES

Later, in Section 4, the significance of the summations (2.7) and (2.8) in representation theory will be manifested.

Irrespective of this representation application, however, each of the two sequences (2.7) and (2.8)—now (3.1) and (3.2)—merits some discussion *per se*. Neither sequence appears in [11].

Write, for convenience,

$$\mathcal{J}_n = \sum_{i=2}^{n} J_i, \quad \mathcal{J}_0 = 0, \ \mathcal{J}_1 = 1 \tag{3.1}$$

and

$$\hat{j}_n = \sum_{i=1}^{n} j_i, \quad \hat{j}_0 = 0. \tag{3.2}$$

Consequently, we have the following tabulation for $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ (in both of which the elements are alternatively odd and even):

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mathcal{J}_n$	0	1	4	9	20	41	84	169	340	681	1364	$\cdots$
$\hat{j}_n$	0	1	6	13	30	61	126	253	510	1021	2046	$\cdots$

(3.3)

Simple detective work readily enables us to spot the recurrences (3.4) and (3.5) in (3.3), which we expect to be modeled on (1.1) and (1.2). As with J_n and j_n in Section 2, we arrange the basic features of $\mathcal{J}_n$ and $\hat{j}_n$ in pairs.

Recurrence relations

$$\mathcal{J}_{n+2} = \mathcal{J}_{n+1} + 2\mathcal{J}_n + 3, \tag{3.4}$$

$$\hat{j}_{n+2} = \hat{j}_{n+1} + 2\hat{j}_n + 5. \tag{3.5}$$

Generating functions

$$\sum_{i=1}^{\infty} \mathcal{J}_i x^{i-1} = (1+2x)(1-2x-x^2+2x^3)^{-1}, \tag{3.6}$$

$$\sum_{i=1}^{\infty} \hat{j}_i x^{i-1} = (1+4x)(1-2x-x^2+2x^3)^{-1}. \tag{3.7}$$

Binet forms

$$\mathcal{J}_n = \frac{J_{n+3}-3}{2} = \frac{2^{n+3}+(-1)^n-9}{6}, \tag{3.8}$$

$$\hat{j}_n = \frac{j_{n+2}-5}{2} = \frac{2^{n+2}+(-1)^n-5}{2}. \tag{3.9}$$

Simson formulas

$$\begin{aligned}\mathcal{J}_{n+1}\mathcal{J}_{n-1} - \mathcal{J}_n^2 &= 2^n\{(-1)^{n-1}-1\}+(-1)^n \\ &= -1 \text{ when } n \text{ is odd,}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\hat{j}_{n+1}\hat{j}_{n-1} - \hat{j}_n^2 &= 2^{n-1}\{9(-1)^{n+1}-5\}+5(-1)^n \\ &= 2^{n+1}-5 \text{ when } n \text{ is odd.}\end{aligned} \tag{3.11}$$

Summations

$$\sum_{i=1}^{n} \mathcal{J}_i = \frac{\mathcal{J}_{n+2}-1-3(n+1)}{2}, \tag{3.12}$$

$$\sum_{i=1}^{n} \hat{j}_i = \frac{\hat{j}_{n+2}-1-5(n+1)}{2}. \tag{3.13}$$

Interrelationships

$$\mathcal{J}_{n+1} + 2\mathcal{J}_{n-1} = \hat{j}_{n+1} - 2, \tag{3.14}$$

$$\hat{j}_{n+1} + 2\hat{j}_{n-1} = 3(3\mathcal{J}_{n-1}+2), \tag{3.15}$$

$$\mathcal{J}_{2n} = 4J_{2n} = 4J_n j_n \quad \text{by (2.9)}, \tag{3.16}$$

$$\mathcal{J}_{2n+1} = 4J_{2n+1} - 3, \tag{3.17}$$

$$\hat{j}_{2n} = 2j_{2n} - 4 = 6J_{2n} \quad \text{by (2.18)}, \tag{3.18}$$

$$\hat{j}_{2n+1} = 2j_{2n+1} - 1, \tag{3.19}$$

$$\mathcal{J}_{n+1} + \mathcal{J}_n = \begin{cases} \hat{j}_{n+1} & n \text{ even}, \\ \hat{j}_{n+1} - 1 & n \text{ odd}. \end{cases} \tag{3.20}$$

$$\hat{j}_{n+1} + \hat{j}_n = 3.2^{n+1} - 5, \tag{3.21}$$

$$\mathcal{J}_n - \mathcal{J}_{n-1} = J_{n+1}, \tag{3.22}$$

$$\hat{j}_n - \hat{j}_{n-1} = j_n, \tag{3.23}$$

$$\mathcal{J}_n - \mathcal{J}_{n-2} = 2^n, \tag{3.24}$$

$$\hat{j}_n - \hat{j}_{n-2} = 3 \cdot 2^{n-1}, \tag{3.25}$$

$$\hat{j}_{n+r} - \hat{j}_{n-r} = \frac{3}{2}(\mathcal{J}_{n+r} - \mathcal{J}_{n-r}) = 6(J_{n+r} - J_{n-r}) = 2(j_{n+r} - j_{n-r}), \tag{3.26}$$

$$\mathcal{J}_{n+2}\mathcal{J}_{n-2} - \mathcal{J}_n^2 = 2^{n-1}\{(-1)^n - 9\}, \tag{3.27}$$

$$\hat{j}_{n+2}\hat{j}_{n-2} - \hat{j}_n^2 = 2^{n-2} \cdot 9\{(-1)^n - 5\}, \tag{3.28}$$

$$\lim_{n\to\infty}\left(\frac{\mathcal{J}_{n+1}}{\mathcal{J}_n}\right) = \lim_{n\to\infty}\left(\frac{\hat{j}_{n+1}}{\hat{j}_n}\right) = 2, \tag{3.29}$$

$$\lim_{n\to\infty}\left(\frac{\hat{j}_n}{\mathcal{J}_n}\right) = \frac{3}{2}, \tag{3.30}$$

$$3\mathcal{J}_n - 2\hat{j}_n = \begin{cases} 0 & n \text{ even}, \\ 1 & n \text{ odd}, \end{cases} \tag{3.31}$$

$$3\mathcal{J}_{2n} = 2\hat{j}_{2n} \quad \text{by (3.31)} \tag{3.32}$$

$$\mathcal{J}_n - J_n = \begin{cases} j_n - 1 & n \text{ odd}, \\ j_n - 2 & n \text{ even}, \end{cases} \tag{3.33}$$

$$\hat{j}_n - J_n = \frac{5}{2}(J_{n+1} - 1), \tag{3.34}$$

$$\mathcal{J}_n - j_n = \begin{cases} J_n - 1 & n \text{ odd}, \\ J_n - 2 & n \text{ even}, \end{cases} \tag{3.35}$$

$$\hat{j}_n - \mathcal{J}_n = J_{n+1} - 1. \tag{3.36}$$

Determinantal evaluations

$$\Delta_{\mathcal{J}} = \begin{vmatrix} \mathcal{J}_n & \mathcal{J}_{n+1} & \mathcal{J}_{n+2} \\ \mathcal{J}_{n+1} & \mathcal{J}_{n+2} & \mathcal{J}_{n+3} \\ \mathcal{J}_{n+2} & \mathcal{J}_{n+3} & \mathcal{J}_{n+4} \end{vmatrix} = 3(-1)^{n+1}2^{n+2} \tag{3.37}$$

and

$$\Delta_{\hat{j}} = \begin{vmatrix} \hat{j}_n & \hat{j}_{n+1} & \hat{j}_{n+2} \\ \hat{j}_{n+1} & \hat{j}_{n+2} & \hat{j}_{n+3} \\ \hat{j}_{n+2} & \hat{j}_{n+3} & \hat{j}_{n+4} \end{vmatrix} = 45(-1)^{n+1}2^{n+1} = \frac{15}{2}\Delta_{\mathcal{J}} \quad \text{by (3.37)}, \tag{3.38}$$

for which are required *inter alia*

$$\mathcal{J}_n \mathcal{J}_{n+3} - \mathcal{J}_{n+1}\mathcal{J}_{n+2} = 2^{n+1}\{(-1)^n - 3\} \tag{3.39}$$

and

$$\hat{j}_n \hat{j}_{n+3} - \hat{j}_{n+1}\hat{j}_{n+2} = 2^n \cdot 3\{3(-1)^n - 5\}. \tag{3.40}$$

With similar notation, it follows obviously from (1.1) and (1.2) that $\Delta_J = \Delta_j = 0$.

Our selection of properties of $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ in (3.4)-(3.40) does not exhaust the many pleasant features of these research-friendly sequences. However, they do give a "flavor" to $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$. It might be noted that, on calculation,

$$\mathcal{J}_n \hat{j}_n \neq \mathcal{J}_{2n}. \tag{3.41}$$

[Because $\{\hat{j}_n\}$ is not a Lucas-type sequence as $\{j_n\}$ is, i.e., $\hat{j}_0 \neq 2$, the "classical" relation of the type (2.9) cannot hold. Indeed, the left-hand side of (3.41) is rather unlovely.] Divisibility properties of (3.16) and (3.18) might also be observed.

Associated Sequences

With notation for *associated sequences* of $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$ similar to that for $\{J_n\}$ and $\{j_n\}$ in (2.30)-(2.33), we derive

$$\mathcal{J}_n^{(1)} = \hat{j}_{n+1} - 2 \quad \text{by (3.14)} \tag{3.42}$$

and

$$\hat{j}_n^{(1)} = 3(3\mathcal{J}_{n-1} + 2) \quad \text{by (3.15)}. \tag{3.43}$$

Invoking (3.14) and (3.15), we have, eventually,

$$\mathcal{J}_n^{(2m)} = 3^{2m}\mathcal{J}_n, \tag{3.44}$$

$$\mathcal{J}_n^{(2m+1)} = 3^{2m}(\hat{j}_{n+1} - 2), \tag{3.45}$$

$$\hat{j}_n^{(2m)} = 3^{2m}\hat{j}_n, \tag{3.46}$$

$$\hat{j}_n^{(2m-1)} = 3^{2m-1}(3\mathcal{J}_{n-1} + 2). \tag{3.47}$$

More briefly,

$$\left.\begin{aligned} \mathcal{J}_n^{(2m)} &= \hat{j}_{n+1}^{(2m-1)} - 2 \cdot 3^{2m-1} \\ \hat{j}_n^{(2m)} &= \mathcal{J}_{n-1}^{(2m+1)} + 2 \cdot 3^{2m} \end{aligned}\right\}. \tag{3.48}$$

Both $\mathcal{J}_n^{(k)}$ and $\hat{j}_n^{(k)}$ are also expressible in terms of J_n and j_n, but this alternative produces slightly less attractive formulas.

Each of the sequences $\{\mathcal{J}_n^{(1)}\}$ and $\{\hat{j}_n^{(1)}\}$ in (3.42) and (3.43) may be regarded as a separate individual entity with a mathematical life of its own, as for $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$, leading *inter alia* to Binet forms, generating functions, Simson formulas, recurrence relations, summation formulas, and miscellaneous interrelationships of varying importance.

Graphs

Suppose we label a pair of rectangular Cartesian axes $\hat{j}$ $(= y)$ and $\mathcal{T}$ $(= x)$. Then (3.30), as n takes on its permissible values, the coordinates $\{\mathcal{T}_n, \hat{j}_n\}$ cluster about the line $y = \frac{3}{2}x$, appearing alternately on opposite sides of this line. Likewise (2.22), in a changed notation, the points (J_n, j_n) as n varies approximate to the line $y = 3x$.

4. JACOBSTHAL REPRESENTATION OF POSITIVE INTEGERS: $\{J_n\}$

Primarily, our concern now is to answer the question: "Can a positive integer N be represented as a sum of Jacobsthal numbers?"

Considerations of minimality and maximality of a representation do not enter into the argument at this stage. Nor does the possibility of uniqueness. Of course, for any minimal representation of N in terms of $\{J_n\}$, we should need

$$N = \sum_{i=2}^{\infty} \Pi_i J_i \quad (\Pi_i = 0, 1, 2) \tag{4.1}$$

subject to the criterion

$$\Pi_i = 2 \Rightarrow \Pi_{i+1} = 0 \tag{4.2}$$

by virtue of (1.1). (Cf. the corresponding Pell condition for minimality [7].)

Why the lower bound $i = 2$ in (4.1)?

Recall from (1.3) that $J_1 = J_2 = 1$. To avoid problems with this two-fold designation of 1, we will omit J_1 from our deliberations and therefore deal only with $\{J_n\}_{n\geq 2}$.

Accordingly, write

$$J'_n = J_{n+1} \tag{4.3}$$

(i.e., $J'_1 = J_2 = 1, \ldots,$ with $J'_0 = 1$) and

$$\Pi'_i = \Pi_{i+1}. \tag{4.4}$$

One has from (2.14), adjusted by (4.3), that

$$2J'_n = J'_{n+1} - 1 < J'_{n+1}, \quad n \text{ odd}, \tag{2.14a}$$

$$2J'_n = J'_{n+1} + 1 > J'_{n+1}, \quad n \text{ even}. \tag{2.14b}$$

For the set $\{S_k\}$ of digits 0, 1, 2 of length k,

$$(\Pi'_1, \Pi'_2, \ldots, \Pi'_k), \tag{4.5}$$

let us use the following symbolism:

$$\left.\begin{aligned} N_k^{\max} &= \text{ the largest integer in } S_k \\ N_k^{\min} &= \text{ the smallest integer in } S_k \\ R_k &= \text{ the range of integers in } S_k \\ I_k &= \text{ the number of integers in } S_k \end{aligned}\right\}. \tag{4.6}$$

Now (Table 2), in each block of k coefficient digits, the smallest number is necessarily given by

$$(0, 0, 0, \ldots, 0, 1) \tag{4.7}$$

i.e.,

$$N_k^{\min} = J'_k \quad \text{by (4.7)}, \tag{4.8}$$

and the largest number by either

$$(0, 0, 0, \ldots, 0, 2), \quad k \text{ odd}, \tag{4.9}$$

or

$$(1, 1, 1, \ldots, 1, 1), \quad k \text{ even}. \tag{4.10}$$

Clearly, then,

$$N_k^{\max} = 2J'_k = J'_{k+1} - 1 \quad \text{by (4.9), (2.14a)}, \quad k \text{ odd}, \tag{4.11}$$

or

$$\begin{aligned} N_k^{\max} &= \sum_{i=1}^{k} J'_i = \mathcal{T}_k \quad \text{by (4.10), (3.1)}, \quad k \text{ even}, \\ &= \frac{J'_{k+2} - 3}{2} \quad \text{by (2.7)} \\ &= J'_{k+1} - 1 \quad \text{by (1.1), (2.14b)}, \end{aligned} \tag{4.12}$$

i.e.,

$$N_k^{\max} = J'_{k+1} - 1 \quad \text{for all } k. \tag{4.13}$$

From (4.8) and (4.13), we derive

$$\begin{aligned} I_k &= (J'_{k+1} - 1) - (J'_k - 1) \quad \text{obviously} \\ &= J'_{k+1} - J'_k \\ &= 2J'_{k-1} \quad \text{by (1.1)}. \end{aligned} \tag{4.14}$$

Thus, by (4.8) and (4.14),

Lemma 1:

$$J'_k \le N \le J'_{k+1} - 1. \tag{4.15}$$

For example, $J'_{10}\ (= 683) \le N = 1{,}000 \le J'_{11} - 1\ (= 1{,}367 - 1 = 1{,}366)$.

Lemma 2: k is uniquely determined by N.

For instance, $N = 1{,}000 \Rightarrow k = 10$.

Therefore, it has been shown that

Theorem 1: Every positive integer N has a representation of the form

$$N = \sum_{i=1}^{\infty} \Pi'_i J'_i \tag{4.16}$$

where $\Pi'_i = 0, 1, 2$, and $\Pi'_i = 2 \Rightarrow \Pi'_{i+1} = 0$.

Details of the discussion encapsulated in Theorem 1 are assembled, in the symbolism of (4.6), in Table 1.

TABLE 1. Data for Representations Involving $\{J_n\}_{n\geq 2}$

k	S_k	R_k	$N_k^{\min}$	$N_k^{\max}$	I_k
1	S_1	1, 2	J_1'	$J_2'-1$	$2\,(=2J_0')$
2	S_2	3, 4	J_2'	$J_3'-1$	$2\,(=2J_1')$
3	S_3	5, ..., 10	J_3'	$J_4'-1$	$6\,(=2J_2')$
4	S_4	11, ..., 20	J_4'	$J_5'-1$	$10\,(=2J_3')$
5	S_5	21, ..., 42	J_5'	$J_6'-1$	$22\,(=2J_4')$
6	S_6	43, ..., 84	J_6'	$J_7'-1$	$42\,(=2J_5')$
⋮	⋮	⋮	⋮	⋮	⋮
k	S_k	$J_k', \ldots, J_{k+1}'$	J_k'	$J_{k+1}'-1$	$2J_{k-1}'$

Specific information for the representation summarized in Table 1 is provided in Table 2. Recall the notation (4.3).

While based on the minimality criterion (4.2), our representation is by no means unique. As a simple illustration, $N=6$ and $N=7$ are also given by the sequences of coefficients (0, 2) and (1, 2), respectively. But our choice of representation of an integer N consistently includes the greatest $J_n' : J_n' \leq N$, e.g., $6 = J_1' + J_3'\,(=1+5)$ rather than $6 = 2J_2'\,(=2\times 3)$ and $7 = 2J_1' + J_3'$ $(=2+5)$ rather than $7 = J_1' + 2J_2'\,(=1+6)$. Infinitely many similar situations exist, along with variations of them.

Our chosen representation in Table 2 has the virtues of simplicity and methodical structure. Because of the usual patterns apparent in Table 2, we may refer to this representation as a *patterned representation*.

For a detailed, but different approach to the representation of integers by means of Jacobsthal numbers, one might consult [1], which investigates a "special" sequence. This sequence is indeed our Jacobsthal sequence, though this cognomen is never alluded to.

5. JACOBSTHAL REPRESENTATION OF POSITIVE INTEGERS: $\{j_n\}$

Turning now to $\{j_n\}$, we may generally parallel the arguments used in Section 4, though here we need to commence the sequence with $j_0\,(=2)$, for otherwise there is no representation possible for the numbers 3 and 4.

Key results corresponding to (2.14a) and (2.14b) are, from (2.14),

$$2j_n = j_{n+1} - 3 < j_{n+1}, \quad n \text{ odd}, \tag{2.14c}$$

and

$$2j_n = j_{n+1} + 3 > j_{n+1}, \quad n \text{ even}. \tag{2.14d}$$

Symbolism used in Section 4 for $\{J_n\}$ will now, for $\{j_n\}$, be replaced by non-capital letters. However, the set $\{s_k\}$ of digits 0, 1, 2 analogous to (4.5) must now become

$$(\pi_0, \pi_1, \pi_2, \ldots, \pi_k), \tag{5.1}$$

which is of length $k+1$.

Adapting the notation in (4.6), we may proceed to establish and arrange the data in Table 3, using methods similar to those in the previous section.

TABLE 2. A Representation of Integers $1 \leq N \leq 100$ by Jacobsthal Numbers J_n

	J_2	J_3	J_4	J_5	J_6	J_7	J_8		J_2	J_3	J_4	J_5	J_6	J_7	J_8
1	1							51		1	1			1	
2	2							52	1	1	1			1	
3		1						53			2			1	
4	1	1						54				1		1	
5			1					55	1			1		1	
6	1		1					56	2			1		1	
7	2		1					57		1		1		1	
8		1	1					58	1	1		1		1	
9	1	1	1					59			1	1		1	
10			2					60	1		1	1		1	
11				1				61	2		1	1		1	
12	1			1				62		1	1	1		1	
13	2			1				63	1	1	1	1		1	
14		1		1				64					1	1	
15	1	1		1				65	1				1	1	
16			1	1				66	2				1	1	
17	1		1	1				67		1			1	1	
18	2		1	1				68	1	1			1	1	
19		1	1	1				69			1		1	1	
20	1	1	1	1				70	1		1		1	1	
21					1			71	2		1		1	1	
22	1				1			72		1	1		1	1	
23	2				1			73	1	1	1		1	1	
24		1			1			74			2		1	1	
25	1	1			1			75				1	1	1	
26			1		1			76	1			1	1	1	
27	1		1		1			77	2			1	1	1	
28	2		1		1			78		1		1	1	1	
29		1	1		1			79	1	1		1	1	1	
30	1	1	1		1			80			1	1	1	1	
31			2		1			81	1		1	1	1	1	
32				1	1			82	2		1	1	1	1	
33	1			1	1			83		1	1	1	1	1	
34	2			1	1			84	1	1	1	1	1	1	
35		1		1	1			85							1
36	1	1		1	1			86	1						1
37			1	1	1			87	2						1
38	1		1	1	1			88		1					1
39	2		1	1	1			89	1	1					1
40		1	1	1	1			90			1				1
41	1	1	1	1	1			91	1		1				1
42					2			92	2		1				1
43						1		93		1	1				1
44	1					1		94	1	1	1				1
45	2					1		95			2				1
46		1				1		96				1			1
47	1	1				1		97	1			1			1
48			1			1		98	2			1			1
49	1		1			1		99		1		1			1
50	2		1			1		100	1	1		1			1

TABLE 3. Data for Representations Involving $\{j_n\}_{n\geq 0}$

k	s_k	r_k	$N_k^{\min}$	$N_k^{\max}$	i_k
1	s_1	$1, \ldots, 4$	j_1	$j_2 - 1$	$4\ (= 2j_0)$
2	s_2	$5, 6$	j_2	$j_3 - 1$	$2\ (= 2j_1)$
3	s_3	$7, \ldots, 16$	j_3	$j_4 - 1$	$10\ (= 2j_2)$
4	s_4	$17, \ldots, 30$	j_4	$j_5 - 1$	$14\ (= 2j_3)$
5	s_5	$31, \ldots, 64$	j_5	$j_6 - 1$	$34\ (= 2j_4)$
6	s_6	$65, \ldots, 126$	j_6	$j_7 - 1$	$62\ (= 2j_5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	s_k	$j_k, \ldots, j_{k+1} - 1$	j_k	$j_{k+1} - 1$	$2j_{k-1}$

Now (Table 4), in each block of $k+1$ coefficient digits, the smallest number must be given by

$$(0, 0, 0, \ldots, 0, 0, 1), \tag{5.2}$$

i.e.,

$$N_k^{\min} = j_k \quad \text{by (5.2)}, \tag{5.3}$$

and the largest by either

$$(1, 0, 0, \ldots, 0, 0, 2), \quad k \text{ odd}, \tag{5.4}$$

or

$$(0, 1, 1, 1, \ldots, 1, 1, 1), \quad k \text{ even}. \tag{5.5}$$

Then

$$\begin{aligned} N_k^{\max} &= 2j_k + 2, \quad k \text{ odd}, \\ &= j_{k+1} - 3 + 2 \quad \text{by (2.14c)} \\ &= j_{k+1} - 1, \end{aligned} \tag{5.6}$$

while

$$\begin{aligned} N_k^{\max} &= \sum_{i=1}^{k} j_i = \hat{j}_k \quad \text{by (3.2)}, \quad n \text{ even}, \\ &= \frac{j_{k+2} - 5}{2} \quad \text{by (2.8)} \\ &= \frac{2j_{k+1} + 3 - 5}{2} \quad \text{by (1.2), (2.14d)} \\ &= j_{k+1} - 1, \end{aligned} \tag{5.7}$$

i.e., for all k,

$$N_k^{\max} = j_{k+1} - 1. \tag{5.8}$$

Thus,

Lemma 3: $j_k \leq N \leq j_{k+1} - 1$.

Lemma 4: k is uniquely determined by N.

Examples: $j_9\ (= 511) \leq N = 1{,}000 \leq j_{10} - 1\ (= 1{,}025 - 1 = 1{,}024)$; $N = 1{,}000 \Rightarrow k = 9$.

TABLE 4. A Representation of Integers $1 \le N \le 100$ by Jacobsthal-Lucas Numbers j_n

	j_0	j_1	j_2	j_3	j_4	j_5	j_6		j_0	j_1	j_2	j_3	j_4	j_5	j_6
1		1						51	1	1			1	1	
2	1							52	1	2			1	1	
3	1	1						53			1		1	1	
4	1	2						54		1	1		1	1	
5			1					55				1	1	1	
6		1	1					56		1		1	1	1	
7				1				57	1			1	1	1	
8		1		1				58	1	1		1	1	1	
9	1			1				59	1	2		1	1	1	
10	1	1		1				60			1	1	1	1	
11	1	2		1				61		1	1	1	1	1	
12			1	1				62						2	
13		1	1	1				63		1				2	
14				2				64	1					2	
15		1		2				65							1
16	1			2				66		1					1
17					1			67	1						1
18		1			1			68	1	1					1
19	1				1			69	1	2					1
20	1	1			1			70			1				1
21	1	2			1			71		1	1				1
22			1		1			72				1			1
23		1	1		1			73		1		1			1
24				1	1			74	1			1			1
25		1		1	1			75	1	1		1			1
26	1			1	1			76	1	2		1			1
27	1	1		1	1			77			1	1			1
28	1	2		1	1			78		1	1	1			1
29			1	1	1			79				2			1
30		1	1	1	1			80		1		2			1
31						1		81	1			2			1
32		1				1		82					1		1
33	1					1		83		1			1		1
34	1	1				1		84	1				1		1
35	1	2				1		85	1	1			1		1
36			1			1		86	1	2			1		1
37		1	1			1		87			1		1		1
38				1		1		88		1	1		1		1
39		1		1		1		89				1	1		1
40	1			1		1		90		1		1	1		1
41	1	1		1		1		91	1			1	1		1
42	1	2		1		1		92	1	1		1	1		1
43			1	1		1		93	1	2		1	1		1
44		1	1	1		1		94			1	1	1		1
45				2		1		95		1	1	1	1		1
46		1		2		1		96						1	1
47	1			2		1		97		1				1	1
48					1	1		98	1					1	1
49		1			1	1		99	1	1				1	1
50	1				1	1		100	1	2				1	1

Theorem 2: Every positive integer N has a representation of the form

$$N = \sum_{i=1}^{\infty} \pi_i j_i , \tag{5.9}$$

where $\pi_i = 0, 1, 2$, and $\pi_i = 2 \Rightarrow \pi_{i+1} = 0$.

Actual details of the j_n-representations are supplied in Table 4 above. As in the case of $\{J_n\}$, these representations contain the criterion for minimality [i.e., condition (4.2) adjusted to π_i], but our chosen representation is nonunique, being selected for convenience to demonstrate that a representation does exist. For instance, we may also have the following representations (cf. Table 5), in which dots denote zeros:

TABLE 5

$N =$ 34	· · · · 2	45	· 1 2 · 2
35	· 1 · · 2	46	1 · 2 · 2
36	1 · · · 2	48	1 1 2 · 2
·	· · · · · ·	48	· · · · 1 1

The tabulation in Table 4 again expresses a *patterned representation*.

6. FINALE

A mild investigation into the possibility of maximum representations was essayed, but no conclusions are offered here. Nevertheless, we reiterate that both $\{J_n\}$ and $\{\hat{j}_n\}$ correspond to the *MinMax sequences* for Pell numbers that were introduced and examined in [7].

Our presentation of some of the basic features of Jacobsthal representations is meant to whet the appetite for further analyses of their properties. Among the opportunities available for exploration are, at least, the following three:

(a) polynomials $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$ which generalize $\{\mathcal{J}_n\}$ and $\{\hat{j}_n\}$,
(b) generalizations of (3.4) and (3.5) when the additive constant is k, and
(c) negatively-subscripted Jacobsthal numbers $\{\mathcal{J}_{-n}\}$ and $\{\hat{j}_{-n}\}$.

Preliminary studies of these topics have been completed by the author, and papers prepared.

For a selection of references relevant to our treatment of representations, one may consult [5]. (Reference [10], though not strictly germane to this paper, is included to remedy an omission in the choice in [5].)

Historical Note

The origins of Jacobsthal numbers (1), 1, 3, 5, 11, 21, ..., where the first term in (1.3) does not occur, predate Jacobsthal's article [9]. Indeed [11], they and their *loi de récurrence* (and Binet form) are traceable, in a trigonometrical setting, to *Nouvelle Correspondance Mathématique*, Vol. 6 (1880), page 146, being there associated with the name of Brocard.

Another, but much later, reference [11] is to page 12 of Vol. 26 (1963) of *Eureka*, the journal of the Archimedeans (Cambridge University Mathematical Society). Here, the first term 1 in (1.3) is given; however, the occurrence of the Jacobsthal numbers is in a purely recreational context, namely: given the first six nonzero terms of (1.3), determine the next two numbers in the sequence.

Jacobsthal polynomials [3], [9] are natural algebraic extensions of their numerical counterparts. Knowing the long history of many mathematical ideas, we should be mildly surprised if the first use of the Jacobsthal numbers did not antedate the year 1880.

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