EXTENSION OF A SYNTHESIS FOR A CLASS OF POLYNOMIAL SEQUENCES

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1. MOTIVATION

Featured in [2] was a pair of generalized functions

$$W_n^{(k)}(x) = \frac{\Delta^k(x)\{\alpha^n(x) - (-1)^k \beta^n(x)\}}{\Delta(x)} \tag{1.1}$$

and

$$\mathcal{W}_n^{(k)}(x) = \Delta^k(x)\{\alpha^n(x) + (-1)^k \beta^n(x)\}, \tag{1.2}$$

where

$$\begin{cases} \alpha(x) = \dfrac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}, \\ \beta(x) = \dfrac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}, \end{cases} \tag{1.3}$$

giving

$$\begin{cases} \alpha(x) + \beta(x) = p(x), \\ \alpha(x)\beta(x) = -q(x), \\ \alpha(x) - \beta(x) = \sqrt{p^2(x) + 4q(x)} = \Delta(x). \end{cases} \tag{1.4}$$

Observe that (1.1) and (1.2) lead to

$$W_0^{(k)}(x) = \Delta^k(x)\frac{\{1 - (-1)^k\}}{\Delta(x)} \tag{1.5}$$

and

$$\mathcal{W}_0^{(k)}(x) = \Delta^k(x)\{1 + (-1)^k\}. \tag{1.6}$$

Hence

$$W_0^{(0)}(x) = 0 \qquad (W_0^{(0)}(0) = 0) \tag{1.7}$$

and

$$\mathcal{W}_0^{(0)}(x) = 2 \qquad (\mathcal{W}_0^{(0)}(0) = 2). \tag{1.8}$$

When $k = 1$, clearly (1.1) and (1.2) in conjunction compress to

$$W_n^{(1)}(x) = \mathcal{W}_n^{(0)}(x) = \mathcal{W}_n(x), \tag{1.9}$$

and

$$\mathcal{W}_n^{(1)}(x) = \Delta^2(x)W_n^{(0)}(x) = \Delta^2(x)W_n(x). \tag{1.10}$$

Generally,

$$W_n^{(k)}(x) = \mathcal{W}_n^{(k-1)}(x). \tag{1.11}$$

Properties of (1.1) and (1.2) were developed in [2] and then related to several special cases of pairs of polynomial sequences, and to numerical particularizations of them arising when $x = 1$ or $x = \frac{1}{2}$ as appropriate.

For a partial description, from a different viewpoint, of some of the material in [2], the reader is referred to [1].

Special cases of (1.1) and (1.2) to which we will refer are [2] by polynomial symbolism and name [with corresponding $p(x)$-value and $q(x)$-value]:

$$\begin{cases} & & & & p(x) & q(x) \\ F_n(x): & \text{Fibonacci} & L_n(x): & \text{Lucas} & x & 1 \\ P_n(x): & \text{Pell} & Q_n(x): & \text{Pell-Lucas} & 2x & 1 \\ J_n(x): & \text{Jacobsthal} & j_n(x): & \text{Jacobsthal-Lucas} & 1 & 2x \\ \mathcal{F}_n(x): & \text{Fermat} & f_n(x): & \text{Fermat-Lucas} & 3x & -2 \\ U_n(x): & \leftarrow \text{Chebyshev} \rightarrow & :T_n(x) & & 2x & -1 \\ \mathcal{U}_n(x): & \leftarrow \text{Hyperbolic} \rightarrow & :\mathcal{V}_n(x) & & 2x & -1 \end{cases} \tag{1.12}$$

For the Chebyshev polynomials, we have $x = \cos\theta$, whereas in the case of the hyperbolic functions we know that $x = \cosh t$.

Toward the end of [2] it was suggested that one of the many extensions of that research was an investigation of the numerical values of (1.1) and (1.2) when k and/or n are negative.

Here, we propose to examine the general theory of polynomials (1.1) and (1.2) for negative k and n. A smooth transition from positive to negative is usually effected. Our endeavors bring into being a collection of Theorems A′, B′, ..., F′ paralleling those in [2]. Of these, the last incorporates the desired synthesis.

2. NEGATIVE SUBSCRIPTS AND SUPERSCRIPTS

Negative Subscripts

After a little calculation using (1.1) and (1.2), we deduce that

$$W_{-n}^{(k)}(x) = -(-1)^k(-q)^{-n}W_n^{(k)}(x) \tag{2.1}$$

and

$$\mathcal{W}_{-n}^{(k)}(x) = (-1)^k(-1)^{-n}\mathcal{W}_n^{(k)}(x), \tag{2.2}$$

showing the connection between positive and negative subscripts.

More particularly, when $k = 0$,

$$W_{-n}(x) = -(-q)^{-n}W_n(x) \tag{2.3}$$

and

$$\mathcal{W}_{-n}(x) = (-q)^{-n}\mathcal{W}_n(x). \tag{2.4}$$

Special Cases

Combining (1.12), (2.1), and (2.2), we derive

$$\begin{cases} F_{-n}^{(k)}(x) = -(-1)^{k-n}F_n^{(k)}(x), & L_{-n}^{(k)}(x) = (-1)^{k-n}L_n^{(k)}(x), \\ P_{-n}^{(k)}(x) = -(-1)^{k-n}P_n^{(k)}(x), & Q_{-n}^{(k)}(x) = (-1)^{k-n}Q_n^{(k)}(x), \\ J_{-n}^{(k)}(x) = -(-1)^{k}(-2x)^{-n}J_n^{(k)}(x), & j_{-n}^{(k)}(x) = (-1)^{k}(-2x)^{-n}j_n^{(k)}(x), \\ \mathcal{F}_{-n}^{(k)}(x) = -(-1)^{k}2^{-n}\mathcal{F}_n^{(k)}(x), & f_{-n}^{(k)}(x) = (-1)^{k}2^{-n}f_n^{(k)}(x), \\ U_{-n}^{(k)}(x) = -(-1)^{k}U_n^{(k)}(x), & T_{-n}^{(k)}(x) = (-1)^{k}T_n^{(k)}(x), \\ \mathcal{U}_{-n}^{(k)}(x) = -(-1)^{k}\mathcal{U}_n^{(k)}(x), & \mathcal{V}_{-n}^{(k)}(x) = (-1)^{k}\mathcal{V}_n^{(k)}(x). \end{cases} \tag{2.5}$$

Putting $k = 0$ in (2.5), we have the standard simplifications [refer to (2.3), (2.4)].

Examples

$$J_{-5}(x) = \frac{4x^2+6x+1}{32x^5} = -\frac{1}{(-2x)^5}J_5(x),$$
$$F_{-3}^{(2)}(x) = x^4+5x^2+4 = (x^2+4)(x^2+1) = F_3^{(2)}(x),$$
$$T_{-4}^{(3)}(x) = -64x(x^2-1)^2(2x^2-1) = -T_4^{(3)}(x) \quad (x = \cos\theta).$$

Differentiation

As in [2], when $k = 0$,

$$\frac{d}{dx}\mathcal{W}_{-n}(x) = \begin{cases} -np'(x)W_{-n}(x) & \text{for } p'(x) \neq 0,\ q'(x) = 0, \\ -nq'(x)W_{-n-1}(x) & \text{for } p'(x) = 0,\ q'(x) \neq 0, \end{cases} \tag{2.6}$$

where the superscript dash (′) denotes differentiation with respect to x.

Thus,

$$\frac{d}{dx}f_{-3}(x) = \frac{d}{dx}\left(\frac{27x^3-18x}{8}\right) = -3.3\frac{(-9x^2+2)}{8} = -3.3\mathcal{F}_{-3}(x),$$

$$\frac{d}{dx}j_{-4}(x) = \frac{d}{dx}\left(\frac{8x^2+8x+1}{16x^4}\right) = \frac{-4.2(4x^2+6x+1)}{32x^5} = -4.2J_{-5}(x).$$

Negative Superscripts

What meaning can be attached to a symbol with a negative superscript? From (1.1), (1.2),

$$W_n^{(-k)}(x) = \Delta^{-k}(x)\frac{\{\alpha^n(x)-(-1)^k\beta^n(x)\}}{\Delta(x)} \tag{2.7}$$

and

$$\mathcal{W}_n^{(-k)}(x) = \Delta^{-k}(x)\{\alpha^n(x)+(-1)^k\beta^n(x)\} \tag{2.8}$$

with obvious extensions when n is replaced by $-n$ (i.e., both subscript and superscript are negative). Refer back to (2.1), (2.2).

For instance,

$$P_2^{(-5)}(x) = (4x^2+2)(4x^2+4)^{-3},$$
$$f_3^{(-4)}(x) = 9x(3x^2-2)(9x^2-8)^{-2}.$$

Some Generalized Products

Without difficulty, one may establish the following multiplicative identities, which were omitted from [2]:

$$W_m^{(h)}(x)W_n^{(k)}(x) = \frac{\Delta^{h+k}(x)}{\Delta^2(x)}\{\alpha^{m+n}(x) + (-1)^{h+k}\beta^{m+n}(x) \\ -(-q(x))^n[(-1)^k\alpha^{m-n}(x) + (-1)^h\beta_{m-n}(x)]\}, \tag{2.9}$$

$$W_m^{(h)}(x)\mathcal{W}_n^{(k)}(x) = \frac{\Delta^{h+k}(x)}{\Delta(x)}\{\alpha^{m+n}(x) - (-1)^{h+k}\beta^{m+n}(x) \\ +(-q(x))^n[(-1)^k\alpha^{m-n}(x) - (-1)^h\beta^{m-n}(x)]\}, \tag{2.10}$$

$$\mathcal{W}_m^{(h)}(x)\mathcal{W}_n^{(k)}(x) = \Delta^{h+k}(x)\{\alpha^{m+n}(x) + (-1)^{h+k}\beta^{m+n}(x) \\ +(-q(x))^n[(-1)^k\alpha^{m-n}(x) + (-1)^k\alpha^{m-n}(x)]\}. \tag{2.11}$$

Various combinations of the above involving $\pm h, \pm k, \pm m, \pm n$ might be investigated. For example, (2.10) with (1.10) leads to

$$W_n^{(k)}(x)\mathcal{W}_n^{(-k)}(x) = W_n^{(-k)}(x)\mathcal{W}_n^{(k)}(x) = W_{2n}(x). \tag{2.12}$$

Another pleasing deduction flows from (2.11), namely,

$$\mathcal{W}_n^{(k)}(x)\mathcal{W}_{-n}^{(-k)}(x) - \mathcal{W}_{-n}^{(k)}(x)\mathcal{W}_n^{(-k)}(x) = 0 \tag{2.13}$$

with a similar conclusion for $W_n^{(k)}(x)$.

Again, applying (2.9) and (2.11) in tandem, we obtain

$$\mathcal{W}_n^{(k)}(x)\mathcal{W}_n^{(-k)}(x) - \Delta^2(x)W_n^{(k)}(x)W_n^{(-k)}(x) = 4(-1)^k(-q(x))^n. \tag{2.14}$$

3. BASIC UNIFYING THEOREMS

Theorems A-F in [2] can now be paralleled. Except that we now use (2.1) and (2.2), of course, the proofs follow those in [2].

Our homologous theorems will be labeled Theorem A′, ..., Theorem F′. Enunciations of these theorems are given below.

Theorem A′: $W_{-n}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) = W_{-2n}^{(2k)}(x)$.

Theorem B′(a): $W_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) + W_{-n}^{(k)}(x)\mathcal{W}_{-m}^{(k)}(x) = 2W_{-(m+n)}^{(2k)}(x)$.

If $m = n$, then Theorem B′(a) reduces to Theorem A′.

Replacing m by $-m$, we derive

Corollary B′(a): $W_m^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x) + W_{-n}^{(k)}(x)\mathcal{W}_m^{(k)}(x) = 2W_{m-n}^{(2k)}(x)$,

$$\begin{cases} = 2W_0^{(2k)}(x) & \text{if } m = n, \\ = 0 & \text{by (1.5).} \end{cases}$$

Theorem B'(b): $\mathcal{W}_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)+\Delta^2(x)W_{-m}^{(k)}(x)W_{-n}^{(k)}(x)=2\mathcal{W}_{-(m+n)}^{(2k)}(x)$.

If $m=n$, then Theorem B'(b) contracts to a sum of squares on the left-hand side. Making the transformation $m\to -m$ gives

Corollary B'(b): $\mathcal{W}_{m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)+\Delta^2(x)W_{m}^{(k)}(x)W_{-n}^{(k)}(x)=2\mathcal{W}_{m-n}^{(2k)}(x)$

$$\begin{cases} =2\mathcal{W}_0^{(2k)}(x) & \text{if } m=n, \\ =4\Delta^{2k}(x) & \text{by } (1.6). \end{cases}$$

Theorem C'(a): $W_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)-W_{-n}^{(k)}(x)\mathcal{W}_{-m}^{(k)}(x)=2(-1)^k(-q(x))^{-n}W_{-(m-n)}^{(2k)}(x)$.

Putting $m=n$ yields the trivial identity $0=0$, by (1.5).
Other considerations are: (i) $m=-n$, (ii) interchange m,n.

Theorem C'(b): $\mathcal{W}_{-m}^{(k)}(x)\mathcal{W}_{-n}^{(k)}(x)-\Delta^2(x)W_{-m}^{(k)}(x)W_{-n}^{(k)}(x)=2(-1)^k(-q(x))^{-n}\mathcal{W}_{-(m-n)}^{(2k)}(x)$.

Variations: (i) $m=n$, (ii) $m\to -m$, (iii) m,n interchanged.

Theorem D': $W_{-n+1}^{(k)}(x)+q(x)W_{-n-1}^{(k)}(x)=\mathcal{W}_{-n}^{(k)}(x)$.

Theorem E': $\mathcal{W}_{-n+1}^{(k)}(x)+q(x)\mathcal{W}_{-n-1}^{(k)}(x)=\Delta^2(x)W_{-n}^{(k)}(x)$.

Illustrations

(A'): $\mathcal{F}_{-2}^{(1)}(x)f_{-2}^{(1)}(x)=\dfrac{-3x(9x^2-8)(9x^2-4)}{16}=\mathcal{F}_{-4}^{(2)}(x)$.

(B'(a)): $F_{-1}^{(2)}(x)L_{-2}^{(2)}(x)+F_{-2}^{(2)}(x)L_{-1}^{(2)}(x)=2(x^2+1)(x^2+4)^2=2F_{-3}^{(4)}(x)$.

(B'(b)): $Q_{-1}^{(1)}(x)Q_{-2}^{(1)}(x)+4(x^2+1)P_{-1}^{(1)}(x)P_{-2}^{(1)}(x)=-16x(x^2+1)(4x^2+3)=2Q_{-3}^{(2)}(x)$.

(C'(a)): $U_{-1}^{(1)}(x)T_{-2}^{(1)}(x)-U_{-2}^{(1)}(x)T_{-1}^{(1)}(x)=-8(x^2-1)=-2U_{1}^{(2)}(x)$.

(C'(b)): $\mathcal{V}_{-1}^{(1)}(x)\mathcal{V}_{-2}^{(1)}(x)-4(x^2-1)\mathcal{U}_{-1}^{(1)}(x)\mathcal{U}_{-2}^{(1)}(x)=-16x(x^2-1)=-2\mathcal{V}_{1}^{(2)}(x)$.

(D'): $\mathcal{F}_{-1}^{(3)}(x)-2\mathcal{F}_{-3}^{(3)}(x)=-\dfrac{3x}{4}(9x-8)^2=f_{-2}^{(3)}(x)$.

(E'): $j_{-2}^{(4)}(x)+2xj_{-4}^{(4)}(x)=\dfrac{(2x+1)(8x+1)^3}{8x^3}=(8x+1)J_{-3}^{(4)}(x)$.

In (C'(a)), $x=\cos\theta(\neq 1)$.
In (C'(b)), $x=\cosh t(\neq 1)$.

4. SYNTHESIS

Elementary algebraic calculations in (1.1), (1.2) when m and n are positive or negative allows us to assert the following synopsis of the relationships connecting $W_n^{(k)}(x)$ and $\mathcal{W}_n^{(k)}(x)$.

Theorem F': For all integers m and n,

$$\begin{cases} W_n^{(2m)}(x) = \mathcal{W}_n^{(2m-1)}(x) = \Delta^{2m}(x)W_n(x), \\ \mathcal{W}_n^{(2m)}(x) = W_n^{(2m+1)}(x) = \Delta^{2m}(x)\mathcal{W}_n(x). \end{cases}$$

Examples

$$F_{-3}^{(6)}(x) = L_{-3}^{(5)}(x) = (x^2+4)^3(x^2+1),$$

$$j_4^{(-4)}(x) = J_4^{(-3)}(x) = (8x^2+8x+1)(8x+1)^{-2},$$

$$f_{-3}^{(-5)}(x) = \mathcal{F}_{-3}^{(-4)}(x) = -\frac{(9x^2-2)(9x^2-8)^{-2}}{8},$$

$$T_{-4}^{(-3)}(x) = U_{-4}^{(-2)}(x) = -x(2x^2-1)(x^2-1)^{-1}.$$

This synthesis extends and complements that presented in [2].

Numerical Specializations

Throughout this paper it is useful to make appropriate numerical substitutions in theory. So,

$$F_{-3}^{(6)}(1) = L_{-3}^{(5)}(1) = 250,$$

$$j_4^{(-4)}(1) = J_4^{(-3)}(1) = \frac{17}{81},$$

$$f_{-3}^{(-5)}(1) = \mathcal{F}_{-3}^{(-4)}(1) = -\frac{7}{8},$$

$$T_{-4}^{(-3)}\left(\frac{1}{2}\right) = U_{-4}^{(-2)}\left(\frac{1}{2}\right) = -\frac{1}{3}.$$

5. A CONCLUDING MISCELLANY

Simson Formulas

Analogs of *Simson's formula* are readily established by means of (1.1), (1.2) for $k > 0$, with immediate extension when $k \to -k$:

$$W_{n+1}^{(k)}(x)W_{n-1}^{(k)}(x) - \{W_n^{(k)}(x)\}^2 = (-1)^{k+1}(-q(x))^{n-1}\Delta^{2k}(x), \tag{5.1}$$

and

$$\mathcal{W}_{n+1}^{(k)}(x)\mathcal{W}_{n-1}^{(k)}(x) - \{\mathcal{W}_n^{(k)}(x)\}^2 = (-1)^{k}(-q(x))^{n-1}\Delta^{2k+2}(x). \tag{5.2}$$

Similar results apply when $n \to -n$.

Variations of these orthodox Simson formulas (Simsonic variations!) include the "*inverted*" *Simson formulas*

$$W_n^{(k+1)}(x)W_n^{(k-1)}(x) - \{W_n^{(k)}(x)\}^2 = 4(-1)^{k}(-q(x))^{n}\Delta^{2k-2}(x), \tag{5.3}$$

and

$$\mathcal{W}_n^{(k+1)}(x)\mathcal{W}_n^{(k-1)}(x) - \{\mathcal{W}_n^{(k)}(x)\}^2 = 4(-1)^{k+1}(-q(x))^{n}\Delta^{2k}(x) \tag{5.4}$$

in which the roles of subscript and superscript in (5.1) and (5.2) have been reversed.

Hybrid Results

Use of (1.1), (1.2) produces the "hybrid Simson formulas"

$$W_{n+1}^{(k)}(x)W_{n-1}^{(k)} - \Delta^{-2}(x)\{\mathcal{W}_n^{(k)}(x)\}^2 = -(-1)^k p^2(x)(-q(x))^{n-1}\Delta^{2k-2}(x), \tag{5.5}$$

and

$$\Delta^{-2}(x)\mathcal{W}_{n+1}^{(k)}(x)\mathcal{W}_{n-1}^{(k)}(x) - \{W_n^{(k)}(x)\}^2 = (-1)^k p^2(x)(-q(x))^{n-1}\Delta^{2k-2}(x). \tag{5.6}$$

Clearly,

$$(5.5)+(5.6)=0 \qquad (i).$$

This is also confirmed by looking at

$$(5.1)+\Delta^{-2}(5.2)=0 \qquad (ii),$$

since the left-hand sides of (i), (ii) are merely re-arrangements of each other.

Further formulas arise when $k \to -k$ and/or $n \to -n$.

Searching for new results involving the data in this paper is an extremely pleasurable activity. Readers may wish to reflect on some of the possibilities.

Surveying the material in this paper and in [2], one is left wondering whether there may be other sets of polynomial-pairs whose major properties may be assembled by means of a synthesis of some kind.

REFERENCES

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