ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-509 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

The *continued fractions (base k)* are defined as follows:

$$[u_1, u_2, \ldots, u_n]_k = u_1 + \frac{k}{u_2+}\frac{k}{u_3+}\cdots\frac{k}{u_n}, \quad n = 1, 2, \ldots, \tag{1}$$

where k is an integer $\neq 0$ and $(u_i)_{i=1}^{\infty}$ is an arbitrary sequence of real numbers.

Given a prime p with $\left(\frac{-k}{p}\right) = 1$ (Legendre symbol) and $k \not\equiv 0 \pmod{p}$, let h be the solution of the congruence

$$h^2 \equiv -k \pmod{p}, \quad \text{with } 0 < h < \tfrac{1}{2}p. \tag{2}$$

Suppose a symmetric continued fraction (base k) exists, such that

$$\frac{p}{h} = [a_1, a_2, \ldots, a_{n+1}, a_{n+1}, \ldots, a_1]_k, \tag{3}$$

where the a_i's are integers, n is even, and $k \mid a_i, i = 2, 4, \ldots, n$. Show that the integers x and y exist, with g.c,d.$(x, y) = 1$, given by

$$\frac{x}{y} = [a_{n+1}, \ldots, a_1]_k \tag{4}$$

which satisfy

$$x^2 + ky^2 = p. \tag{5}$$

H-510 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Pell numbers by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Show that, for $n = 1, 2, \ldots$,

$$P_n = \sum_{k \in A_n} (-1)^{[(3k-2n-1)/4]} 2^{[3k/2]} \binom{n+k}{2k+1},$$

where [ ] denotes the greatest integer function and

$$A_n = \{k \in \{0, 1, \ldots, n-1\} \mid 3k \not\equiv 2n \pmod 4\}.$$

H-511 ***Proposed by M. N. Deshpande, Aurangabad, India***

Find all possible pairs of positive integers m and n such that $m(m+1) = n(m+n)$. [Two such pairs are: $m = 1, n = 1$; and $m = 9, n = 6$.]

H-512 ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***

The *Fibonacci pseudoprimes* (or FPP's) are those composite n with g.c.d.$(n, 10) = 1$ such that $n \mid F_{n-\varepsilon_n}$ where ε_n is the Jacobi symbol $\left(\frac{5}{n}\right)$. Suppose $n = p(p+2)$, where p and $p+2$ are "twin primes." Prove that n is a FPP if and only if $p \equiv 7 \pmod{10}$.

H-508 (Corrected) ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the Fibonacci polynomials by $F_0(x) = 0$, $F_1(x) = 1$, $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$, for $n \geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y) = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^k F_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1) = n\sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x) = n\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k}, \quad x \neq 0; \tag{3}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2 = n\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x) = (2n-1)\sum_{k=0}^{2n-2} \frac{(-1)^2}{k+1}\binom{2n+k-1}{2k+1}x^k F_{k+1}(4/x). \tag{6}$$

SOLUTIONS

Probably

H-493 ***Proposed by Stefano Mascella and Piero Filipponi, Rome, Italy***
(Vol. 33, no. 1, February 1995)

Let $P_k(d)$ denote the probability that the k^{th} digit (from left) of an ℓ digit ($\ell \geq k$) Fibonacci number F_n (expressed in base 10) whose subscript is randomly chosen within a *large* interval $[n_1, n_2]$ $(n_1 \gg n_2)$ is d.

That the sequence $\{F_n\}$ obeys Benford's law is a well-known fact (e.g., see [1] and [2]). In other words, it is well known that $P_1(d) = \log_{10}(1+1/d)$.

Find an expression for $P_2(d)$.

References

1. P. Filipponi. "Some Probabilistic Aspects of the Terminal Digits of Fibonacci Numbers." *The Fibonacci Quarterly* (to appear).
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981):175-77.

Solution by Norbert Jensen, Kiel, Germany

Let $d \in \{0, 1, \ldots, 9\}$. For each $i \in \mathbb{N}$, let $A_{i,d}$ be the set of those $n \in \mathbb{N}$ for which $F_n \geq 10^{i-1}$ and the i^{th} digit (from the left) of F_n equals d. For all $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$, let $I(n_1, n_2)$ denote the set of all integers n with $n_1 \leq n \leq n_2$. Let $p := \log_{10}((1+1/(1+d\cdot 10^{-1}))(1+1/(2+d\cdot 10^{-1}))\ldots (1+1/(9+d\cdot 10^{-1})))$.

Let $n_1 \in \mathbb{N}$. We show that

$$\frac{|A_{2\cdot d} \cap I(n_1, n_2)|}{|I(n_1, n_2)|} \to p \text{ as } n_2 \text{ tends to infinity.}$$

This proves that $P_2(d)$ is approximately equal to p for a given interval $I(n_1, n_2)$, provided that n_2 is large enough.

[Note that, in general, it is **not** true that $P_2(d) = p$ for all $d \in \{0, 1, \ldots, 9\}$ for a finite interval $I(n_1, n_2)$ with a certain minimum of members. If we had one, we could add $n_2 + 1$ to it. Suppose, without loss of generality, that the second digit of F_{n_2+1} is $\neq d$. Then

$$|A_{2\cdot d} \cap I(n_1, n_2+1)| < p|I(n_1, n_2+1)|.$$

A similar argument applies to $P_1(d)$ and $\log_{10}(1+1/d)$.]

Proof: Step (0). $\log_{10}(\alpha)$ is irrational.

Proof: Suppose it is rational. Then we find $a \in \mathbb{Z}, b \in \mathbb{N}$ such that $\log_{10}(\alpha) = a/b$. Hence, $\log_{10}(\alpha^b) = b\cdot\log_{10}(\alpha) = a$ and $F_b\alpha + F_{b-1} = \alpha^b = 10^a$, whence $\sqrt{5} \in \mathbb{Q}$, a contradiction. Q.E.D. Step (0).

Step (1). $\log_{10}(F_n) = n\cdot\log_{10}(\alpha) + \log_{10}(1-(-1)^n\beta^{2n}) - \log_{10}(\sqrt{5})$ for all $n \in \mathbb{N}$.

Proof:

$$F_n = (\alpha^n - \beta^n)/\sqrt{5} = \alpha^n(1-(\beta/\alpha)^n)/\sqrt{5} = \alpha^n(1-(-\beta^2)^n)/\sqrt{5} = \alpha^n(1-(-1)^n\beta^{2n})/\sqrt{5}.$$

Q.E.D. Step (1).

For any $x \in \mathbb{R}$, let $\langle x\rangle$ denote the purely fractional part of x, i.e., $\langle x\rangle = x - [x]$.

Step (2). The sequence $(\langle\log_{10}(F_n)\rangle)$ is uniformly distributed modulo 1.

Proof: By (0) and according to Example 2.1 on page 8 of [1], the sequence $(\langle n\log_{10}(\alpha)\rangle)$ is uniformly distributed modulo 1. Since $\log_{10}(1-(-1)^n\beta^{2n})$ converges (to zero), the sequence $(\langle n\log_{10}(\alpha) + \log_{10}(1-(-1)^n\beta^{2n}) - \log_{10}\sqrt{5}\rangle)$ is uniformly distributed (see [1], Theorem 1.2, p. 3). Thus, $(\langle\log_{10}(F_n)\rangle)$ is uniformly distributed modulo 1 by Step (1). Q.E.D. Step (2).

Step (3). Let $Z_1 \in \{1, 2, \ldots, 9\}$, $Z_2 \in \{0, 1, \ldots, 9\}$. Let $n \in \mathbb{N}$. Let $t = [\log_{10}(F_n)]$. We have the following equivalences:

$\Leftrightarrow$ There is an $R \in \mathbb{N}_0$ with $R < 10^{t-1}$ such that $F_n = Z_1 \cdot 10^t + Z_2 \cdot 10^{t-1} + R$.

$\Leftrightarrow$ There is an $R \in \mathbb{N}_0$ with $R < 10^{t-1}$ such that

$$\langle \log_{10}(F_n) \rangle = \log_{10}(F_n) - [\log_{10}(F_n)] = \log_{10}(Z_1 + Z_2 \cdot 10^{-1} + R \cdot 10^{-t}).$$

$\Leftrightarrow$ $\langle \log_{10}(F_n) \rangle \in [\log_{10}(Z_1 + Z_2 \cdot 10^{-1}), \log_{10}(Z_1 + (Z_2 + 1) \cdot 10^{-1})]$.

So, by the definition of "uniform distribution" ([1], p. 1), we have that

$$\frac{|A_{1,Z_1} \cap A_{2,Z_2} \cap I(n_1, n_2)|}{|I(n_1, n_2)|}$$

converges to the length of the interval $[\log_{10}(Z_1 + Z_2 \cdot 10^{-1}), \log_{10}(Z_1 + (Z_2 + 1) \cdot 10^{-1})]$, namely, to $\log_{10}(1 + 1/(Z_1 + Z_2 \cdot 10^{-1}))$, when n_2 tends to infinity. Since the intervals are disjoint for different pairs of digits (Z_1, Z_2), it is clear that we can fix $Z_2 = d$ and take the sum over $Z_1 = 1, 2, \ldots, 9$. Q.E.D.

Remarks:

1. The above proof can be abridged by using Washington's theorem [2] for the base $b = 10^2$.

2. We even have the following more general result: For each $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that, for all $n_1 \in \mathbb{N}$ and all $n_2 \in \mathbb{N}$ with $n_2 \geq n_1 + n_0$, we have

$$\left| \frac{|A_{2,d} \cap I(n_1, n_2)|}{|I(n_1, n_2)|} - p \right| < \varepsilon.$$

In other words: We have uniform convergence. The quality of the approximation depends only on the cardinality of $|I(n_1, n_2)|$, not on the choice of n_1.

Proof of Remark 2: By Weyl's criterion, the sequence $(\langle n \log_{10}(\alpha) \rangle)$ is well distributed modulo 1 (see [1], p. 40, p. 42, Example 5.2). This implies that $(\langle \log_{10}(F_n) \rangle)$ is well distributed (see [1], Theorem 5.4, p. 43). Modifying the arguments of (3) with respect to n_1, we obtain the assertion. Q.E.D.

References

1. L. Kuipers & H. Niederreiter. *Uniform Distribution of Sequences*. New York, 1974.
2. L. C. Washington. "Benford's Law for Fibonacci and Lucas Numbers." *The Fibonacci Quarterly* **19.2** (1981).

Also solved by P. Bruckman.

Apparently

H-494 ***Proposed by David M. Bloom, Brooklyn College, New York, NY (Vol. 33, no. 1, February 1995)***

It is well known that if $P(p)$ is the Fibonacci entry point ("rank of apparition") of the odd prime $p \neq 5$, then $P(p)$ divides $p + e$ where $e = \pm 1$. In [1] it is stated without proof [Theorem 5(b)] that the integer $(p + e) / P(p)$ has the same parity as $(p - 1) / 2$. Give a proof.

Reference

1. D. Bloom. "On Periodicity in Generalized Fibonacci Sequences." *Amer. Math. Monthly* **72** (1965):856-61.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that $\varepsilon = -(5/p)$, where $(5/p)$ denotes Legendre's symbol. In 1930, D. H. Lehmer (see [1], p. 325, Lemma 5) proved that

$$p \mid F_{(p+\varepsilon)/2} \text{ if and only if } p \equiv 1 \pmod 4. \tag{1}$$

Let $k = (p+\varepsilon)/P(p)$. If k is even, then $p \mid F_{(k/2)P(p)} = F_{(p+\varepsilon)/2}$, since $P(p) \mid (k/2)P(p)$ and $p \mid F_{P(p)}$. Thus, we have $p \equiv 1 \pmod 4$, by (1), so that $k \equiv 0 \equiv (p-1)/2 \pmod 2$. Now, suppose that k is odd. Assuming that $p \equiv 1 \pmod 4$, we would have $p \mid F_{(p+\varepsilon)/2} = F_{kP(p)/2}$, again by (1). This would imply that $P(p)$ is even, that $k \ge 3$, and that $p \mid L_{P(p)/2}$, since p divides $F_{P(p)} = F_{P(p)/2}L_{P(p)/2}$, but does not divide $F_{P(p)/2}$. Now, from

$$F_{kP(p)/2} = L_{P(p)/2}F_{(k-1)P(p)/2} - (-1)^{P(p)/2}F_{(k-2)P(p)/2},$$

it then would follow that $p \mid F_{(k-2)P(p)/2}$. Repeating this argument, we would arrive at the contradiction that $p \mid F_{P(p)/2}$. Thus, we must have $p \equiv 3 \pmod 4$, so that $k \equiv 1 \equiv (p-1)/2 \pmod 2$. This completes the solution.

Reference

1. Lawrence Somer. "The Divisibility Properties of Primary Lucas Recurrences with Respect to Primes." *The Fibonacci Quarterly* **18.4** (1980):316-34.

Also solved by P. Bruckman, A. Dujella, N. Jensen, and the proposer.

Achieve Parity

<u>H-495</u> ***Proposed by Paul S. Bruckman, Salmiya, Kuwait***
(Vol. 33, no. 1, February 1995)

Let p be a prime $\neq 2, 5$ and let $Z(p)$ denote the *Fibonacci entry-point* of p (i.e., the smallest positive integer m such that $p \mid F_m$). Prove the following "Parity Theorem" for the Fibonacci entry-point:

A. If $p \equiv 11$ or $19 \pmod{20}$, then $Z(p) \equiv 2 \pmod 4$;
B. If $p \equiv 13$ or $17 \pmod{20}$, then $Z(p)$ is odd;
C. If $p \equiv 3$ or $7 \pmod{20}$, then $4 \mid Z(p)$.

Solution by the proposer

We employ two well-known results, stated as lemmas without proof.

Lemma 1: If $p \neq 2, 5$ and $p' = \frac{1}{2}\left(p - \left(\frac{5}{p}\right)\right)$, then (i) $p \mid F_{p'}$ if $p \equiv 1 \pmod 4$, or (ii) $p \mid L_{p'}$ if $p \equiv -1 \pmod 4$.

An equivalent formulation of Lemma 1 is restated as

Lemma 1': If $p \neq 2, 5$ and $q = \frac{1}{2}(p-1)$, then (i) $p \mid F_q$ if $p \equiv 1$ or $9 \pmod{20}$; (ii) $p \mid L_q$ if $p \equiv 11$ or $19 \pmod{20}$; (iii) $p \mid F_{q+1}$ if $p \equiv 13$ or $17 \pmod{20}$; (iv) $p \mid L_{q+1}$ if $p \equiv 3$ or $7 \pmod{20}$.

Lemma 2: $Z(p)$ is even for all primes $p > 2$ if and only if $p \mid L_n$ for some n.

Lemma 2 implies that if $p > 2$ and $p \mid L_n$, then $Z(p) = 2n/r$ for some odd integer r dividing n.

Proof of A: By Lemma 1'(ii), $p|L_q$. Then $Z(p)|2q$ and $Z(p)$ must be even, by Lemma 2. Since $2q = p-1 \equiv 2 \pmod 4$ in this case, it follows that $Z(p) \equiv 2 \pmod 4$.
Proof of B: By Lemma 1'(iii), $p|F_{q+1}$. Then $Z(p)|(q+1)$. In this case, $q+1 = \frac{1}{2}(p+1) \equiv 7$ or $9 \pmod{10}$, an odd integer. Therefore, $Z(p)$ must be odd.
Proof of C: By Lemma 1'(iv), $p|L_{q+1}$. Then $Z(p) = 2(q+1)/r = (p+1)/r$, where r is odd, and $r|(p+1)$. Since $p+1 \equiv 0 \pmod 4$ in this case, we see that $4|Z(p)$.

Note: No inference may be made about the parity of $Z(p)$ if $p \equiv 1$ or $9 \pmod{20}$.

Also solved by D. Bloom, A. Dujella, N. Jensen, and H.-J. Seiffert.

FLUPPS and ELUPPS

H-496 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 33, no. 2, May 1995)

Let n be a positive integer >1 with $\text{g.c.d.}(n,10)=1$, and $\delta = (5/n)$, a Jacobi symbol. Consider the following congruences:
(1) $F_{n-\delta} \equiv 0 \pmod n$, $L_n \equiv 1 \pmod n$;
(2) $F_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$ if $n \equiv 1 \pmod 4$, $L_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$ if $n \equiv 3 \pmod 4$.
Composite n which satisfy (1) are called *Fibonacci-Lucas pseudoprimes,* abbreviated "FLUPPS." Composite n which satisfy (2) are called *Euler-Lucas pseudoprimes with parameters* $(1,-1)$, abbreviated "ELUPPS." Prove that FLUPPS and ELUPPS are equivalent.

Solution by Andrej Dujella, University of Zagreb, Croatia

(1) $\Rightarrow$ (2): It is easy to check that, for $\delta \in \{-1,1\}$, it holds: $2L_n - 5F_{n-\delta} = \delta L_{n-\delta}$. Considering that, from (1), it follows that $L_{n-\delta} \equiv 2\delta \pmod n$. From the identity $L_{2n} + 2\cdot(-1)^n = L_n^2$ [see S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section* (Chichester: Halsted, 1989), (17c)], we have $L^2_{\frac{1}{2}(n-\delta)} = L_{n-\delta} + 2\cdot(-1)^{\frac{1}{2}(n-\delta)} \equiv 2\delta + 2\cdot(-1)^{\frac{1}{2}(n-\delta)} \pmod n$.

If $n \equiv 3 \pmod 4$, then $2\delta + 2\cdot(-1)^{(n-\delta)/2} = 2\delta + 2\cdot(-1)^{(1+\delta)/2} = 0$; therefore, $L_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$.

If $n \equiv 1 \pmod 4$, then $2\delta + 2\cdot(-1)^{(n-\delta)/2} = 2\delta + 2\cdot(-1)^{(1+\delta)/2} = 4\delta$, and using $\text{g.c.d.}(F_m, L_m) \le 2$ and $F_{n-\delta} = F_{(n-\delta)/2}L_{(n-\delta)/2} \equiv 0 \pmod n$, we have $F_{\frac{1}{2}(n-\delta)} \equiv 0 \pmod n$.

(2) $\Rightarrow$ (1): From $F_{n-\delta} = F_{(n-\delta)/2}L_{(n-\delta)/2}$ and (2), it follows that $F_{n-\delta} \equiv 0 \pmod n$. Now, from $2L_n - 5F_{n-\delta} = \delta L_{n-\delta}$ it may be concluded that $2\delta L_n \equiv L_{n-\delta} \pmod n$.

If $n \equiv 3 \pmod 4$, we have $2\delta L_n \equiv L^2_{\frac{1}{2}(n-\delta)} - 2\cdot(-1)^{\frac{1}{2}(1+\delta)} \equiv 2\cdot(-1)^{\frac{1}{2}(1+\delta)} \equiv 2\delta \pmod n$; therefore, $L_n \equiv 1 \pmod n$.

If $n \equiv 1 \pmod 4$, we have $2\delta L_n \equiv 5F^2_{\frac{1}{2}(n-\delta)} + 2\cdot(-1)^{\frac{1}{2}(1-\delta)} \equiv 2\cdot(-1)^{\frac{1}{2}(1-\delta)} \equiv 2\delta \pmod n$, and again $L_n \equiv 1 \pmod n$.

Also solved by A. G. Dresel, H.-J. Seiffert, and the proposer.

Editorial Note: *The editor will appreciate it if all proposals and solutions are submitted in typed format.*

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