CLASSES OF IDENTITIES FOR THE GENERALIZED FIBONACCI NUMBERS $G_n = G_{n-1} + G_{n-c}$ FROM MATRICES WITH CONSTANT VALUED DETERMINANTS

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The generalized Fibonacci numbers $\{G_n\}$, $G_n = G_{n-1} + G_{n-c}$, $n \ge c$, $G_0 = 0$, $G_1 = G_2 = \cdots = G_{c-1} = 1$, are the sums of elements found on successive diagonals of Pascal's triangle written in left-justified form, by beginning in the left-most column and moving up $(c-1)$ and right 1 throughout the array [1]. Of course, $G_n = F_n$, the n^{th} Fibonacci number, when $c = 2$. Also, $G_n = u(n-1; c-1, 1)$, where $u(n; p, q)$ are the generalized Fibonacci numbers of Harris and Styles [2]. In this paper, elementary matrix operations make simple derivations of entire classes of identities for such generalized Fibonacci numbers, and for the Fibonacci numbers themselves.

1. INTRODUCTION

Begin with the sequence $\{G_n\}$, such that

$$G_n = G_{n-1} + G_{n-3},\ n \ge 3,\ G_0 = 0,\ G_1 = G_2 = 1. \tag{1.1}$$

For the reader's convenience, the first values are listed below:

n	0	1	2	3	4	5	6	7	8	9	10
G_n	0	1	1	1	2	3	4	6	9	13	19

n	11	12	13	14	15	16	17	18	19	20	21
G_n	28	41	60	88	129	189	277	406	595	872	1278

These numbers can be generated by a simple scheme from an array which has 0, 1, 1 in the first column, and which is formed by taking each successive element as the sum of the element above and the element to the left in the array, except that in the case of an element in the first row we use the last term in the preceding column and the element to the left:

$$\begin{array}{ccccccc} 0 & 1 & 4 & 13 & [41]\downarrow & 129 & \cdots \\ 1 & 2 & 6 & [19]\rightarrow & \underline{60} & 189 & \cdots \\ 1 & 3 & 9 & 28 & 88 & 277 & \cdots \end{array} \tag{1.2}$$

If we choose a 3×3 array from any three consecutive columns, the determinant is 1. If any 3×4 array is chosen with 4 consecutive columns, and row reduced by elementary matrix methods, the solution is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{pmatrix}. \tag{1.3}$$

We note that, in any row, any four consecutive elements d, e, f, and g are related by

$$d - 3e + 4f = g. \tag{1.4}$$

Each element in the third row is one more than the sum of the $3k$ elements in the k preceding columns; i.e., $9 = (3+2+1+1+1+0)+1$. Each element in the second row satisfies a "column property" as the sum of the three elements in the preceding column; i.e., $60 = 28+19+13$ or, alternately, a "row property" as each element in the second row is one more than the sum of the element above and all other elements in the first row; i.e., $60 = (41+13+4+1+0)+1$. Each element in the third row is the sum of the element above and all other elements to the left in the second row; i.e., $28 = 19+6+2+1$. It can be proved by induction that

$$G_1 + G_2 + G_3 + \cdots + G_n = G_{n+3} - 1, \tag{1.5}$$

$$G_3 + G_6 + G_9 + \cdots + G_{3k} = G_{3k+1} - 1, \tag{1.6}$$

$$G_1 + G_4 + G_7 + \cdots + G_{3k+1} = G_{3k+2}, \tag{1.7}$$

$$G_2 + G_5 + G_8 + \cdots + G_{3k+2} = G_{3k+3}; \tag{1.8}$$

which compare with

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1, \tag{1.9}$$

$$F_1 + F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2}, \tag{1.10}$$

$$F_2 + F_4 + F_6 + \cdots + F_{2k} = F_{2k+1} - 1, \tag{1.11}$$

for Fibonacci numbers.

The reader should note that forming a two-rowed array analogous to (1.2) by taking 0, 1 in the first column yields Fibonacci numbers, while taking 0, 1, 1, ..., with an infinite number of rows, forms Pascal's triangle in rectangular form, bordered on the top by a row of zeros. We also note that all of these sequences could be generated by taking the first column as all 1's or as 1, 2, 3, ..., or as the appropriate number of consecutive terms in the sequence. They all satisfy "row properties" and "column properties." The determinant and matrix properties observed in (1.2) and (1.3) lead to entire classes of identities in the next section.

2. IDENTITIES FOR THE FIBONACCI NUMBERS AND FOR THE CASE $c = 3$

Write a 3×3 matrix $A_n = (a_{ij})$ by writing three consecutive terms of $\{G_n\}$ in each column and taking $a_{11} = G_n$, where $c = 3$ as in (1.1):

$$A_n = \begin{pmatrix} G_n & G_{n+p} & G_{n+q} \\ G_{n+1} & G_{n+p+1} & G_{n+q+1} \\ G_{n+2} & G_{n+p+2} & G_{n+q+2} \end{pmatrix}. \tag{2.1}$$

We can form matrix A_{n+1} by applying (1.1), replacing row 1 by (row 1 + row 3) in A_n followed by two row exchanges, so that

$$\det A_n = \det A_{n+1}. \tag{2.2}$$

Let $n = 1, p = 1, q = 2$ in (2.1) and find $\det A_1 = -1$. Thus, $\det A_n = -1$ for

$$A_n = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} \\ G_{n+1} & G_{n+2} & G_{n+3} \\ G_{n+2} & G_{n+3} & G_{n+4} \end{pmatrix}. \tag{2.3}$$

As another special case of (2.1), use 9 consecutive elements of $\{G_n\}$ to write

$$A_n = \begin{pmatrix} G_n & G_{n+3} & G_{n+6} \\ G_{n+1} & G_{n+4} & G_{n+7} \\ G_{n+2} & G_{n+5} & G_{n+8} \end{pmatrix}, \tag{2.4}$$

which has $\det A_n = 1$.

These simple observations allow us to write many identities for $\{G_n\}$ effortlessly. We illustrate our procedure with an example. Suppose we want an identity of the form

$$\alpha G_n + \beta G_{n+1} + \gamma G_{n+2} = G_{n+4}.$$

We write an augmented matrix A_n^*, where each column contains three consecutive elements of $\{G_n\}$ and where the first row contains G_n, G_{n+1}, G_{n+2}, and G_{n+4}:

$$A_n^* = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} & G_{n+4} \\ G_{n+1} & G_{n+2} & G_{n+3} & G_{n+5} \\ G_{n+2} & G_{n+3} & G_{n+4} & G_{n+6} \end{pmatrix}.$$

Then take a convenient value for n, say $n = 1$, and use elementary row operations on the augmented matrix A_1^*,

$$A_1^* = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

to obtain a generalization of the "column property" of the introduction,

$$G_n + G_{n+1} + G_{n+2} = G_{n+4}, \tag{2.5}$$

which holds for any n.

While we are using matrix methods to solve the system

$$\begin{cases} \alpha G_n + \beta G_{n+1} + \gamma G_{n+2} = G_{n+4}, \\ \alpha G_{n+1} + \beta G_{n+2} + \gamma G_{n+3} = G_{n+5}, \\ \alpha G_{n+2} + \beta G_{n+3} + \gamma G_{n+4} = G_{n+6}, \end{cases}$$

notice that each determinant that would be used in a solution by Cramer's rule is of the form $\det A_n = \det A_{n+1}$ from (2.1) and (2.2), and, moreover, the determinant of coefficients equals -1 so that there will be integral solutions. Alternately, by (1.1), notice that (α, β, γ) will be a solution of $\alpha G_{n+3} + \beta G_{n+4} + \gamma G_{n+5} = G_{n+7}$ whenever (α, β, γ) is a solution of the system above for any $n \geq 0$ so that we solve all such equations whenever we have a solution for any three consecutive values of n.

We could make one identity at a time by augmenting A_n with a fourth column beginning with G_{n+w} for any pleasing value for w, except that $w < 0$ would force extension of $\{G_n\}$ to negative subscripts. However, it is not difficult to solve

$$A_n^* = \begin{pmatrix} G_n & G_{n+1} & G_{n+2} & G_{n+w} \\ G_{n+1} & G_{n+2} & G_{n+3} & G_{n+w+1} \\ G_{n+2} & G_{n+3} & g_{n+4} & G_{n+w+2} \end{pmatrix}$$

by taking $n = 0$ and elementary row reduction, since $G_{w+2} - G_{w+1} = G_{w-1}$ by (1.1), and

$$A_0^* = \begin{pmatrix} 0 & 1 & 1 & G_w \\ 1 & 1 & 1 & G_{w+1} \\ 1 & 1 & 2 & G_{w+2} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_w \\ 1 & 0 & 0 & G_{w-2} \\ 0 & 0 & 1 & G_{w-1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & G_{w-2} \\ 0 & 1 & 0 & G_{w-3} \\ 0 & 0 & 1 & G_{w-1} \end{pmatrix},$$

so that

$$G_{n+w} = G_{w-2}G_n + G_{w-3}G_{n+1} + G_{w-1}G_{n+2}. \tag{2.6}$$

For the Fibonacci numbers, we can use the matrix A_n,

$$A_n = \begin{pmatrix} F_n & F_{n+q} \\ F_{n+1} & F_{n+q+1} \end{pmatrix},$$

for which $\det A_n = (-1)\det A_{n+1}$. Of course, when $q = 1$, $\det A_n = (-1)^{n+1}$ where, also, $\det A_n = F_nF_{n+2} - F_{n+1}^2$, giving the well-known

$$(-1)^{n+1} = F_nF_{n+2} - F_{n+1}^2. \tag{2.7}$$

Solve the augmented matrix A_n^* as before,

$$A_n^* = \begin{pmatrix} F_n & F_{n+1} & F_{n+w} \\ F_{n+1} & F_{n+2} & F_{n+w+1} \end{pmatrix},$$

by taking $n = -1$,

$$A_{-1}^* = \begin{pmatrix} 1 & 0 & F_{w-1} \\ 0 & 1 & F_w \end{pmatrix},$$

to obtain

$$F_{w-1}F_n + F_wF_{n+1} = F_{n+w}. \tag{2.8}$$

Identities of the type $\alpha G_n + \beta G_{n+2} + \gamma G_{n+4} = G_{n+6}$ can be obtained as before by row reduction of

$$A_n^* = \begin{pmatrix} G_n & G_{n+2} & G_{n+4} & G_{n+6} \\ G_{n+1} & G_{n+3} & G_{n+5} & G_{n+7} \\ G_{n+2} & G_{n+4} & G_{n+6} & G_{n+8} \end{pmatrix}.$$

If we take $n = 0$, $\det A_0 = 1$, and we find $\alpha = 1, \beta = 2, \gamma = 1$, so that

$$G_{n+6} = G_n + 2G_{n+2} + G_{n+4}. \tag{2.9}$$

In a similar manner, we can derive

$$G_{n+9} = G_n - 3G_{n+3} + 4G_{n+6}, \tag{2.10}$$

$$G_{n+12} = G_n - 2G_{n+4} + 5G_{n+8}, \tag{2.11}$$

where we compare (1.4) and (2.10).

For the Fibonacci numbers, solve

$$A_n^* = \begin{pmatrix} F_n & F_{n+2} & F_{n+4} \\ F_{n+1} & F_{n+3} & F_{n+5} \end{pmatrix}$$

by taking $n = 1$,

$$A_1^* = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix},$$

so that

$$F_{n+4} = -F_n + 3F_{n+2}. \tag{2.12}$$

Similarly,

$$F_{n+6} = F_n + 4F_{n+3}, \tag{2.13}$$

$$F_{n+8} = -F_n + 7F_{n+4}. \tag{2.14}$$

In the Fibonacci case, we can solve directly for F_{n+2p} from

$$A_n^* = \begin{pmatrix} F_n & F_{n+p} & F_{n+2p} \\ F_{n+1} & F_{n+p+1} & F_{n+2p+1} \end{pmatrix}$$

by taking $n = -1$,

$$A_{-1}^* = \begin{pmatrix} 1 & F_{p-1} & F_{2p-1} \\ 0 & F_p & F_{2p} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & (F_pF_{2p-1} - F_{p-1}F_{2p}) / F_p \\ 0 & 1 & F_{2p} / F_p \end{pmatrix} \to \begin{pmatrix} 1 & 0 & (-1)^{p-1} \\ 0 & 1 & L_p \end{pmatrix},$$

since $F_pF_{2p-1} - F_{p-1}F_{2p} = (-1)^{p-1}F_p$ and $F_{2p} = F_pL_p$ are known identities for the Fibonacci and Lucas numbers. Thus,

$$F_{n+2p} = (-1)^{p-1}F_n + L_pF_{n+p}. \tag{2.15}$$

Returning to (2.9), we can derive identities of the form $\alpha G_n + \beta G_{n+2} + \gamma G_{n+4} = G_{n+2w}$ from (2.1) with $p = 2, q = 4$, taking the augmented matrix A_n^* with first row containing G_n, G_{n+2}, G_{n+4}, G_{n+2w}. It is computationally advantageous to take $n = -1$; notice that we can define $G_{-1} = 0$. We make use of $G_{2w} - G_{2w-1} = G_{2w-3}$ from (1.1) to solve

$$A_{-1}^* = \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 1 & 2 & G_{2w} \\ 1 & 1 & 3 & G_{2w+1} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 0 & 1 & G_{2w} - G_{2w-1} \\ 1 & 0 & 1 & G_{2w+1} - G_{2w} \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & G_{2w-1} \\ 0 & 0 & 1 & G_{2w-3} \\ 1 & 0 & 1 & G_{2w-2} \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & 0 & G_{2w-1} - G_{2w-3} \\ 0 & 0 & 1 & G_{2w-3} \\ 1 & 0 & 0 & G_{2w-2} - G_{2w-3} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & G_{2w-5} \\ 0 & 1 & 0 & G_{2w-1} - G_{2w-3} \\ 0 & 0 & 1 & G_{2w-3} \end{pmatrix},$$

obtaining

$$G_{n+2w} = G_{2w-5}G_n + (G_{2w-1} - G_{2w-3})G_{n+2} + G_{2w-3}G_{n+4}. \tag{2.16}$$

In the Fibonacci case, taking $n = -1$,

$$A_n^* = \begin{pmatrix} F_n & F_{n+2} & F_{n+2w} \\ F_{n+1} & F_{n+3} & F_{n+2w+1} \end{pmatrix} \to A_{-1}^* = \begin{pmatrix} 1 & 1 & F_{2w-1} \\ 0 & 1 & F_{2w} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -F_{2w-2} \\ 0 & 1 & F_{2w} \end{pmatrix},$$

we have

$$F_{n+2w} = -F_{2w-2}F_n + F_{2w}F_{n+2}. \tag{2.17}$$

Returning to (2.6) and (2.16), the same procedure leads to

$$G_{n+3w} = G_{3w-6}G_n + (G_{3w} - 4G_{3w-3})G_{n+3} + G_{3w-3}G_{n+6}. \tag{2.18}$$

The Fibonacci case, derived by taking $n = -1$,

$$A_n^* = \begin{pmatrix} F_n & F_{n+3} & F_{n+3w} \\ F_{n+1} & F_{n+4} & F_{n+3w+1} \end{pmatrix},$$

gives us

$$F_{n+3w} = F_nF_{3w-3} / 2 + F_{n+3}F_{3w} / 2, \tag{2.19}$$

where $F_{3m} / 2$ happens to be an integer for any m. Note that $\det A_n = (-1)^{n+1}2$, and hence, $\det A_n \neq \pm 1$. We cannot make a pleasing identity of the form $\alpha G_n + \beta G_{n+4} + \gamma G_{n+8} = G_{n+4w}$ for arbitrary w because $\det A_n \neq \pm 1$, leading to nonintegral solutions. However, we can find an identity for $\{G_n\}$ analogous to (2.15). We solve

$$\begin{cases} \alpha G_{-1} + \beta G_{p-1} + \gamma G_{2p-1} = G_{3p-1}, \\ \alpha G_0 + \beta G_p + \gamma G_{2p} = G_{3p}, \\ \alpha G_1 + \beta G_{p+1} + \gamma G_{2p+1} = G_{3p+1}, \end{cases}$$

for (α, β, γ) by Cramer's rule. Note that the determinant of coefficients D is given by $D = G_{2p}G_{p-1} - G_pG_{2p-1}$. Then $\alpha = A / D$, where A is the determinant

$$A = \begin{vmatrix} G_{3p-1} & G_{p-1} & G_{2p-1} \\ G_{3p} & G_p & G_{2p} \\ G_{3p+1} & G_{p+1} & G_{2p+1} \end{vmatrix}.$$

After making two column exchanges in A, we see from (2.1) and (2.2) that $A = D$, so $\alpha = 1$. Then $\beta = B / D$, where B is the determinant

$$B = \begin{vmatrix} 0 & G_{3p-1} & G_{2p-1} \\ 0 & G_{3p} & G_{2p} \\ 1 & G_{3p+1} & G_{2p+1} \end{vmatrix} = G_{2p}G_{3p-1} - G_{3p}G_{2p-1}.$$

Similarly, $\gamma = C / D$, where $C = G_3G_{p-1} - G_pG_{3p-1}$. Thus,

$$G_{n+3p} = G_n + G_{n+p}(G_{3p-1}G_{2p} - G_{3p}G_{2p-1}) / D + G_{n+2p}(G_{3p}G_{p-1} - G_{3p-1}G_p) / D,$$

where $D = (G_{2p}G_{p-1} - G_pG_{2p-1})$. The coefficients of G_{n+p} and G_{n+2p} are integers for $p = 1, 2, \ldots, 9$, and it is conjectured that they are always integers.

As an observation before going to the general case, notice that identities such as (2.9), (2.10), and (2.11) generate more matrices with constant valued determinants. For example, (2.9) leads to matrix B_n,

$$B_n = \begin{pmatrix} G_n & G_{n+p} & G_{n+q} \\ G_{n+2} & G_{n+p+2} & G_{n+q+2} \\ G_{n+4} & G_{n+p+4} & G_{n+q+4} \end{pmatrix},$$

where $\det B_n = \det B_{n+2}$.

3. THE GENERAL CASE: $G_n = G_{n-1} + G_{n-c}$

The general case for $\{G_n\}$ is defined by

$$G_n = G_{n-1} + G_{n-c},\ n \geq c, \text{ where } G_0 = 0,\ G_1 = G_2 = \cdots = G_{c-1} = 1. \tag{3.1}$$

To write the elements of $\{G_n\}$ simply, use an array of c rows with the first column containing 0 followed by $(c-1)$ 1's, noting that 1, 2, 3, ..., c will appear in the second column, analogous to the array of (1.2). Take each term to be the sum of the term above and the term to the left, where we drop below for elements in the first row as before. Any $c \times c$ array formed from any c consecutive columns will have a determinant value of ± 1. Each element in the c^{th} row is one more than the sum of the ck elements in the k preceding columns, i.e.,

$$G_1 + G_2 + G_3 + \cdots + G_{ck} = G_{c(k+1)} - 1, \tag{3.2}$$

which can be proved by induction. It is also true that

$$G_1 + G_2 + G_3 + \cdots + G_n = G_{n+c} - 1. \tag{3.3}$$

Each array satisfies the "column property" of (2.5) in that each element in the $(c-1)^{\text{st}}$ row is the sum of the c elements in the preceding column and, more generally, for any n,

$$G_{n+c-2} = G_{n-c} + G_{n-(c-1)} + \cdots + G_{n-2} + G_{n-1} \quad (c \text{ terms}). \tag{3.4}$$

Each array has "row properties" such that each element in the i^{th} row, $3 \leq i \leq c$, is the sum of the element above and all other elements to the left in the $(i-1)^{\text{st}}$ row, while each element in the second row is one more than the sum of the elements above and to the left in the first row, or

$$G_0 + G_c + G_{2c} + G_{3c} + \cdots + G_{ck} = G_{ck+1} - 1, \tag{3.5}$$

$$G_m + G_{c+m} + G_{2c+m} + \cdots + G_{ck+m} = G_{ck+m+1},\ m = 1, 2, \ldots, c-1, \tag{3.6}$$

for a total of c related identities reminiscent of (1.6), (1.7), and (1.8).

The matrix properties of Section 2 also extend to the general case. Form the $c \times c$ matrix $A_{n,c} = (a_{ij})$, where each column contains c consecutive elements of $\{G_n\}$ and $a_{11} = G_n$. Then, as in the case $c = 3$,

$$\det A_{n,c} = (-1)^{c-1} \det A_{n+1,c}, \tag{3.7}$$

since each column satisfies $G_{n+c} = G_{n+c-1} + G_n$. We can form $A_{n+1,c}$ from $A_{n,c}$ by replacing row 1 by (row 1 + row c) followed by $(c-1)$ row exchanges.

When we take the special case in which the first row of $A_{n,c}$ contains c consecutive elements of $\{G_n\}$, then $A_{n,c} = \pm 1$. The easiest way to justify this result is to observe that (3.1) cam be used to extend $\{G_n\}$ to negative subscripts. In fact, in the sequence $\{G_n\}$ extended by recursion (3.1), $G_1 = 1$ and G_1 is followed by $(c-1)$ 1's and preceded by $(c-1)$ 0's. If we write the first row of $A_{n,c}$ as $G_n, G_{n-1}, G_{n-2}, \ldots, G_{n-(c-1)}$, then, for $n=1$, the first row is 1, 0, 0, ..., 0. If each column contains c consecutive increasing terms of $\{G_n\}$, then G_n appears on the main diagonal in every row. Thus, $A_{1,c}$ has 1's everywhere on the main diagonal with 0's everywhere above, so that det $A_{1,c} = 1$. That $\det A_{n,c} = \pm 1$ is significant, however, because it indicates that we can write identities following the same procedures as for $c=3$, expecting integral results when solving systems as before. Note that $\det A_{n,c} = \pm 1$ if the first row contains c consecutive elements of $\{G_n\}$, but order does not matter. Also, we have the interesting special case that $\det A_{n,c} = \pm 1$ whenever $A_{n,c}$ contains c^2 consecutive terms of $\{G_n\}$, taken in either increasing or decreasing order, $c \geq 2$. Det $A_{n,c} = 0$ only if two elements in row 1 are equal, since any c consecutive germs of G_n are relatively prime [2].

Again, solving an augmented matrix $A^*_{n,c}$ will make identities of the form

$$G_{n+w} = \alpha_0 G_n + \alpha_1 G_{n+1} + \alpha_2 G_{n+2} + \cdots + \alpha_{c-1} G_{n+c-1}$$

for different fixed values of c, or other classes of identities of your choosing. As examples, we have:

$c=2 \qquad F_{n+w} = F_n F_{w-1} + F_{n+1} F_w,$

$c=3 \qquad G_{n+w} = G_n G_{w-2} + G_{n+1} G_{w-3} + G_{n+2} G_{w-1},$

$c=4 \qquad G_{n+w} = G_n G_{w-3} + G_{n+1} G_{w-4} + G_{n+2} G_{w-5} + G_{n+3} G_{w-2},$

$c=5 \qquad G_{n+w} = G_n G_{w-4} + G_{n+1} G_{w-5} + G_{n+2} G_{w-6} + G_{n+3} G_{w-7} + G_{n+4} G_{w-3},$

...

$c=c \qquad G_{n+w} = G_n G_{w-c+1} + G_{n+1} G_{w-c} + G_{n+2} G_{w-c-1} + \cdots + G_{n+c-1} G_{w-c+2};$

$c=2 \qquad F_{n+3} = F_n + F_{n+1} + F_{n+1},$

$c=3 \qquad G_{n+4} = G_n + G_{n+1} + G_{n+2},$

$c=4 \qquad G_{n+5} = G_n + G_{n+1} + G_{n+3},$

$c=5 \qquad G_{n+6} = G_n + G_{n+1} + G_{n+4},$

...

$c=c \qquad G_{n+c+1} = G_n + G_{n+1} + G_{n+c-1}.$

So many identities, so little time!

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