ON REPETITIONS IN FREQUENCY BLOCKS OF THE GENERALIZED FIBONACCI SEQUENCE $u(3, 1)$ WITH $u_0 = u_1 = 1$

Gyula Darvasi

Teacher's Training College, Department of Mathematics, H-4400 Nyiregyháza, Hungary

Mihály Nagy

Teacher's Training College, Department of Computer Science, H-4400 Nyiregyháza, Hungary

(*Submitted August 1994*)

Let $u_0 = u_1 = 1$ and define the generalized Fibonacci sequence $(u_n) = u(3, 1)$ to satisfy the recurrence relation $u_n = 3u_{n-1} + u_{n-2}$ for $n \geq 2$. For an integer $m > 1$, let $(\overline{u}_n)$ denote the sequence (u_n) considered modulo m. It is known that $(\overline{u}_n)$ is purely periodic [7], that is, there exists a positive integer r such that $\overline{u}_{n+r} = \overline{u}_n$ for all $n = 0, 1, \ldots$. Define $h(m)$ to be the length of a shortest period of $(\overline{u}_n)$, and $S(m)$ to be the set of residue frequencies within any full period of $(\overline{u}_n)$, as well as $A(m, d)$ to denote the number of times the residue d appears in a full period of $(\overline{u}_n)$ ([5], [6]). Hence, for a fixed m, the range of $A(m, d)$ is the set $S(m)$, that means

$$\{A(m, d) : 0 \leq d < m\} = S(m).$$

We say (u_n) is uniformly distributed modulo m if all residues modulo m occur with the same frequency in any full period. In this case, the length of any period will be a multiple of m; moreover, $|S(m)| = 1$ and $A(m, d)$ is a constant function [4].

For a fixed $m \geq 2$, form a number block $B_m \in N^m$ to consist of the frequency values of the residue d when d runs through the complete residue system modulo m. This number block, B_m, will be called the *frequency block* modulo m, which has properties like $(qB_m)^r = q(B_m)^r$ and $((B_m)^r)^s = (B_m)^{rs}$ with

$$(B_m)^r = (\underbrace{B_m, \ldots, B_m}_{r \text{ times}})$$

and $q, r, s \in N$. Here are some examples for B_m together with the period length $h(m)$.

$B_2 = (1, 2)$	$h(2) = 3$
$B_3 = (0, 1, 0)$	$h(3) = 1$
$B_4 = (1, 3, 1, 1)$	$h(4) = 6$
$B_5 = (0, 3, 3, 3, 3)$	$h(5) = 12$
$B_8 = (0, 3, 0, 1, 2, 3, 2, 1)$	$h(8) = 12$
$B_9 = 2(B_3)^3$	$h(9) = 6$
$B_{16} = (0, 3, 0, 1, 2, 3, 0, 1, 0, 3, 0, 1, 2, 3, 4, 1)$	$h(16) = 24$
$B_{18} = (0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0)$	$h(18) = 6$
$B_{26} = 4(B_2)^{13}$	$h(26) = 156$
$B_{27} = 2(B_3)^9$	$h(27) = 18$
$B_{32} = (B_{16})^2$	$h(32) = 48$
$B_{52} = 2(B_4)^{13}$	$h(52) = 156$

$B_{54} = (B_{18})^3$ $\qquad h(54) = 18$

$B_{65} = (B_5)^{13}$ $\qquad h(65) = 156$

$B_{81} = 2(B_3)^{27}$ $\qquad h(81) = 54$

All but the first few examples show a certain kind of repetition in the frequency blocks, that means such frequency blocks can be produced by repetition of their first few elements a whole number of times. For a given m, this repetition is possible only in the case for which there exists an integer $1 < c < m$ such that $c|m$ and $h(c)|h(m)$. Moreover, the first few repeating elements of B_m are the elements of B_c or some multiple of them. Letting $0 \le x < c$, $0 \le y < m$, and $y \equiv x \pmod{c}$, this fact can be expressed by $A(m, y) = q \cdot A(c, x)$ for some positive integer q. A similar result in connection with the uniform distribution was found in [3] for the Fibonacci numbers. The considered sequence (u_n) is uniformly distributed modulo 13^k for $k \ge 1$ (see [1]). Thus, the above examples show that the repetition in the frequency blocks does not occur exclusively in connection with the uniform distribution.

To search for repetition possibilities in the frequency blocks of the sequence (u_n), we made a computer run for moduli $m \le 1000$. However, we did not consider moduli m with $13|m$ because we wanted to investigate the repetition possibilities that had no direct connections with the uniform distribution.

Making use of the above-mentioned notation $A(m, y) = q \cdot A(c, x)$ with $0 \le x < c$, $0 \le y < m$, $y \equiv x \pmod{c}$, and $1 \le q \in N$, we discovered the following:

D1: $A(3^{k+1}, y) = 2 \cdot A(3, x)$ for $k \ge 1$.

D2: $A(3^k c, y) = A(c, x)$ for $k \ge 1$ and $c \in \{18, 21, 33, 36, 45, 51, 57, 69, 72, 87, 90, 93, 111, 123, 126, 144, 147, 159, 180, 198, 201, 219, 231, 237, 252, 291, 303, 306, 315, 321, 327\}$.

D3: $A(p^{k+1}, y) = A(p, x)$ for $k \ge 1$ and $p \in \{11, 17, 29\}$.

D4: $A(11c, y) = A(c, x)$ for $c \in \{22, 33, 44, 55, 66, 77, 88\}$.

D5: $A(17c, y) = A(c, x)$ for $c \in \{34, 51\}$.

D6: $A(2c, y) = A(c, x)$ for $c \in \{16, 48, 144, 368\}$.

D7: $A(6c, y) = A(c, x)$ for $c = 144$.

Now it is natural to ask how the above discoveries could be proved. We will give proofs for some of them.

We note that in this paper (a, b) and $[a, b]$ will denote the greatest common divisor and the least common multiple of the integers a and b, respectively.

Lemma 1: The sequence (u_n) is purely periodic mod 3^r with the exact period length $h(3^r) = 1$ for $r = 1$ and $h(3^r) = 2 \cdot 3^{r-1}$ for $r > 1$. Let w be a fixed integer with $0 \le w < h(3^r)$. If u_w leaves the remainder x mod 3^r $(0 \le x < 3^r)$, then the numbers $u_{w+jh(3^r)}$ $(0 \le j \le 2)$ leave the remainders $x + i \cdot 3^r$ $(0 \le i \le 2)$ mod 3^{r+1} in a certain ordering.

Proof: The fact that (u_n) is purely periodic mod 3^r with period length $h(3^r) = 1$ if $r = 1$ and $h(3^r) = 2 \cdot 3^{r-1}$ if $r > 1$ follows by arguments similar to those given by Wall in Theorems 1, 4, 5, 10, and 12 of [7]. The remainder of Lemma 1 follows from results in the preprint "Bounds for Frequencies of Residues in Second-Order Recurrences Modulo p^r" by Lawrence Somer.

Lemma 2: For $2 \le c \in N$, $(c, 3) = 1$, and $1 \le k \in N$, let $q = h(3^{k+1}c)/3h(3^k c)$. Then $q = 1/3$, $2/3$, or 1 if $k = 1$, and $q = 1/3$ or 1 if $k > 1$.

Proof: Since $(c, 3) = 1$, we have

$$q = \frac{[h(3^{k+1}), h(c)]}{3[h(3^k), h(c)]}.$$

The case $k = 1$ yields

$$q = \frac{[h(9), h(c)]}{3[h(3), h(c)]} = \frac{[6, h(c)]}{3[1, h(c)]} = \frac{2}{(6, h(c))} = \begin{cases} 2 & \text{if } (6, h(c)) = 1, \\ 1 & \text{if } (6, h(c)) = 2, \\ 2/3 & \text{if } (6, h(c)) = 3, \\ 1/3 & \text{if } (6, h(c)) = 6. \end{cases}$$

Now, using the known facts that $h(2) = 3, h(3) = 1, h(6) = 3,$ and $h(c)$ is even for $c > 3$ and $c \ne 6$, we obtain $(6, h(c)) = 1$ iff $c = 1$ or $c = 3$, which are excluded in Lemma 2. Moreover, if $(c, 3) = 1$, then $(6, h(c)) = 3$ iff $c = 2$.

In the case $k > 1$, we have by Lemma 1 that

$$q = \frac{[2 \cdot 3^k h(3), h(c)]}{3[2 \cdot 3^{k-1} h(3), h(c)]} = \frac{[2 \cdot 3^k, h(c)]}{3[2 \cdot 3^{k-1}, h(c)]}$$

$$= \frac{(2 \cdot 3^{k-1}, h(c))}{(2 \cdot 3^k, h(c))} = \begin{cases} 1/3 & \text{if } 3^k | h(c), \\ 1 & \text{if } 3^{t-1} | h(c) \text{ and } 3^t \nmid h(c), \text{ where } 1 \le t \le k. \end{cases}$$

For some $1 \le b \in N$, let $v_3(b)$ denote the exact power of 3 such that $3^{v_3(b)} | b$ but $3^{v_3(b)+1} \nmid b$.

Corollary 1: For $2 \le c \in N$, $(c, 3) = 1$, and $1 \le k \in N$, $q = h(3^{k+1}c)/3h(3^k c)$ is an integer iff $v_3[h(c)] \le k - 1$. In this case, the only possible value for q is $q = 1$.

Corollary 2: For $2 \le c = 3^r s \in N$, $r \in N$, $1 \le s \in N$, $(s, 3) = 1$, we have:

$$r = 0 \Rightarrow h(3c) = h(c),$$

$$r = 1 \Rightarrow h(3c) = \begin{cases} 6h(c) & \text{if } s = 1, \\ 3h(c) & \text{if } s > 2 \text{ and } 3 \nmid h(s), \\ 2h(c) & \text{if } s = 2, \\ h(c) & \text{if } s > 2 \text{ and } 3 | h(s). \end{cases}$$

$$r > 1 \Rightarrow h(3c) = \begin{cases} h(c) & \text{if } 3^r | h(s), \\ 3h(c) & \text{otherwise.} \end{cases}$$

Hence, the value of $q = h(3c)/3h(c)$ with $c = 3^r s$, $r \in N$, $1 \le s \in N$, and $(s, 3) = 1$ cannot be an integer if $r = 0$, or if $r = 1$ and $s \ge 2$ and $3 | h(s)$, or if $r > 1$ and $s > 2$ and $3^r | h(s)$. These cases can be omitted from here on.

Corollary 3: For $2 < c = 3^r s \in N$, $1 \le r$, $s \in N$, and $(s, 3) = 1$, $q = h(3c)/3h(c)$ is an integer iff $r \ge 1$ and $v_3[h(s)] \le r - 1$. Now suppose that q is an integer. If $r = 1$ and $s = 1$, then $q = 2$; if $r = 1$, $s > 1$, and $(s, 3) = 1$, then $q = 1$; if $r > 1$, $s \ge 1$, and $(s, 3) = 1$, then again $q = 1$.

Theorem 1: For $2 < c = 3^r s \in N$, $1 \le r$, $s \in N$, $(s, 3) = 1$, $v_3[h(s)] \le r-1$, and $q = h(3c)/3h(c)$, we have $B_{3c} = q(B_c)^3$.

Proof: **Case 1.** $r = 1$

Now $c = 3s$, $(s, 3) = 1$, and $v_3[h(s)] \le 0 \Rightarrow 3 \nmid h(s)$.

If $s = 1$, then $q = h(9)/3h(3) = 2$. Thus, $B_{3^2} = 2(B_3)^3$ can be checked by computation.

If $s > 1$, then $q = h(3c)/3h(c) = 1$. Thus, we need to prove that $B_{3^2 s} = (B_{3s})^3$.

Since $h(3c) = 3h(c)$ and $h(3s) = h(s)$, we need to show that, for any $w \in N$ and $j \in \{0, 1, 2\}$, the three values of $u_{w+jh(s)}$ are pairwise different modulo 9, and hence also modulo $9s$. Let $j_1, j_2 \in \{0, 1, 2\}$ with $1 \le |j_1 - j_2| \le 2$. For a fixed $w \in N$, let z_1 and z_2 be the residues of the numbers $w + j_1 h(s)$ and $w + j_2 h(s)$ mod $h(9)$, respectively. This means $0 \le |z_1 - z_2| < h(9) = 6$. The consequence of $s > 1$ and $3 \nmid h(s)$ is that $s \ge 7$; therefore, $h(s)$ is even. This yields

$$2 \le h(s) \le h(s)|j_1 - j_2| = |z_1 - z_2| \not\equiv 0 \pmod 6,$$

so that z_1 and z_2 are different mod 6 and, in addition, are not consecutive numbers; whence, u_{z_1} (mod 9) and u_{z_2} (mod 9) also have two different values that can be checked using the following table:

n	0	1	2	3	4	5	6	$\cdots$
u_n (mod 9)	1	1	4	4	7	7	1	$\cdots$

Case 2. $r > 1$

Now $c = 3^r s$, $(s, 3) = 1$, and $v_3[h(s)] \le r - 1 \Rightarrow q = h(3c)/3h(c) = 1$. Thus, we must prove that $B_{3c} = (B_c)^3$.

We need to show that, for any fixed $w \in N$ and $j \in \{0, 1, 2\}$, the numbers $u_{w+jh(c)}$ are pairwise different modulo $3c$. Since $(s, 3) = 1$ and $v_3[h(s)] \le r-1$, we have $h(c) = h(3^r s) = [h(3^r), h(s)] = h(3^r)z$ with some $1 \le z \in n$ and $3 \nmid z$. Hence, for a fixed $w \in N$ and $j \in \{0, 1, 2\}$, the numbers $w + jh(c)$ and $w + jh(3^r)$ are always in the same residue class modulo $h(3^r)$. Therefore, the numbers $u_{w+jh(c)}$ and $u_{w+jh(3^r)}$ are also in the same residue class modulo 3^r. But the numbers $u_{w+jh(3^r)}$ are pairwise different mod 3^{r+1} because of Lemma 1. Thus, the numbers $u_{w+jh(c)}$ are again pairwise different mod 3^{r+1}, and thereby also mod $3c$.

Theorem 2: For $1 \le k \in N$ and $q = h(3^{k+1})/3h(3^k)$, we have $B_{3^{k+1}} = q(B_{3^k})^3$.

Proof: We proceed by induction on k. For $k = 1$, we go back to Case 1 of Theorem 1, whence $q = 2$ and $B_{3^2} = 2(B_3)^3$.

Assume the statement is true for $k > 1$. As a consequence of Case 2 of Theorem 1, we can take $q = 1$. Thus, $B_{3^{(k+1)+1}} = B_{3(3^{k+1})} = 1 \cdot (B_{3^{k+1}})^3 = (q(B_{3^k})^3)^3 = q((B_{3^k})^3)^3 = q(B_{3^{k+1}})^3$.

Corollary 4: For $1 \le k \in N$ and $q = h(3^{k+1})/3h(3^k)$, we have $B_{3^{k+1}} = 2(B_3)^{3^k}$.

Proof: $k = 1 \Rightarrow q = 2$ and $B_{3^2} = 2(B_3)^3$. $k > 1 \Rightarrow q = 1$ and $B_{3^{k+1}} = (B_{3^k})^3 = (B_{3(3^{k-1})})^3 = ((B_{3^{k-1}})^3)^3 = (B_{3^{k-1}})^{3^2} = \cdots = (B_{3^2})^{3^{k-1}} = (2(B_3)^3)^{3^{k-1}} = 2(B_3)^{3^k}$.

Corollary 5: For any $1 \le k \in N$, we have $|S(3^k)| = |S(3)| = 2$.

Thus, we have a complete proof for D1. The statement in D2 is a direct consequence of Theorem 1. The proof of D7 can be done using D2 and D6 as follows:

$$B_{6c} = B_{3(2c)} = (B_{2c})^3 = ((B_c)^2)^3 = (B_c)^6.$$

The proofs of the other discoveries can, for the most part, be carried out in a similar manner, so they are left to the interested reader.

The only reason for considering the above specific problem was Corollary 2 in [1], where it was proved that the sequences $u(3, 1)$ with $u_0 = 1$ and $u_1 \in \{1,3,5\}$ are uniformly distributed mod 13^k for all $k \ge 1$. The reader should consider the related more general sequences $u(p, 1)$ satisfying the recursion relation $u_n = pu_{n-1} + u_{n-2}$ for $n \ge 2$ with $u_0 = u_1 = 1$ and p a fixed odd prime. It can be proved by similar methods that $B_{pk+1} = 2(B_p)^{p^k}$ is also valid for these recurrences; here, B_p refers to the frequency block defined above. The reader might consider proving this result, and possibly other results similar to those found in this paper. In the meantime, it is advisable to remember the fundamental fact that the recurrences $u(p, 1)$ with $u_0 = u_1 = 1$ are irregular modulo p, that is, the vectors (u_0, u_1) and (u_1, u_2) are linearly dependent modulo p.

ACKNOWLEDGMENT

The authors are very grateful to the anonymous referee for some valuable suggestions which improved the clarity of the presentation of this paper.

REFERENCES

1. P. Bundschuh & J.-S. Shiue. "Solution of a Problem on the Uniform Distribution of Integers." *Atti della Accademia Nazionale dei Lincei* **55** (1973):172-77.
2. P. Bundschuh. "On the Distribution of Fibonacci Numbers." *Tamkang J. Math.* **5.1** (1974): 75-79.
3. E. Jacobson. "The Distribution of Residues of Two-Term Recurrence Sequences." *The Fibonacci Quarterly* **28.3** (1990):227-29.
4. E. Jacobson. "A Brief Survey on Distribution Questions for Second-Order Linear Recurrences." *Proceedings of the First Meeting of the Canadian Number Theory Association*, pp. 249-54. Ed. R. A. Mollin and W. de Gruyter. Canada, 1990.
5. A. Schinzel. "Special Lucas Sequences, Including the Fibonacci Sequence, Modulo a Prime." In *A Tribute to Paul Erdös*, pp. 349-57. Ed. A. Baker, B. Bollobás, and A. Hajnal. Cambridge: Cambridge University Press, 1990.
6. L. Somer. "Distribution of Residues of Certain Second-Order Linear Recurrences Modulo *p*—II. *The Fibonacci Quarterly* **29.1** (1991):72-78.
7. D. D. Wall. "Fibonacci Series Modulo *m*." *Amer. Math. Monthly* **67** (1960):525-32.

AMS Classification Number: 11B39

❖❖❖