GENERALIZED FIBONACCI NUMBERS AND THE PROBLEM OF DIOPHANTUS

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(Submitted July 1994)

1. INTRODUCTION

Let n be an integer. A set of positive integers $\{a_1, \ldots, a_m\}$ is said to have *the property of Diophantus of order n*, symbolically $D(n)$ if, for all $i, j = 1, \ldots, m$, $i \neq j$, the following holds: $a_i a_j + n = b_{ij}^2$, where b_{ij} is an integer. The set $\{a_1, \ldots, a_m\}$ is called *a Diophantine m-tuple*.

In this paper we construct several Diophantine quadruples whose elements are represented using generalized Fibonacci numbers. It is a generalization of the following statements (see [8], [12], [6]): The sets

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \quad \text{and} \quad \{n, n+2, 4n+4, 4(n+1)(2n+1)(2n+3)\}$$

have the property $D(1)$; the set

$$\{2F_{n-1}, 2F_{n+1}, 2F_n^3 F_{n+1}F_{n+2}, 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2 - F_n^2)\}$$

has the property $D(F_n^2)$ for all positive integers n.

These results are applied to the Pell numbers and are used to obtain explicit formulas for quadruples with the property $D(\ell^2)$, where ℓ is an integer.

2. PRELIMINARIES

2.1 The Problem of Diophantus

The Greek mathematician Diophantus of Alexandria noted that the numbers x, $x+2$, $4x+4$, and $9x+6$, where $x = 1/16$, have the following property: the product of any two of them increased by 1 is a square of a rational number (see [3]). The French mathematician Pierre de Fermat first found a set with the property $D(1)$, and it was $\{1, 3, 8, 120\}$. Later, Davenport and Baker [2] showed that if there is a set $\{1, 3, 8, d\}$ with the property $D(1)$, then d has to be 120.

In [5], the problem of the existence of Diophantine quadruples with the property $D(n)$ was considered for an arbitrary integer n. The following result was proved: if an integer n is not of the form $4k+2$ and $n \notin \{3, 5, 8, 12, 20, -1, -3, -4\}$, then there exists a quadruple with the property $D(n)$.

Nonexistence of Diophantine quadruples with the property $D(4k+2)$ was proved in [1] and [5].

The sets with the property $D(\ell^2)$ were particularly discussed in [5]. It was proved that for any integer ℓ and any set $\{a, b\}$ with the property $D(\ell^2)$, where ab is not a perfect square, there exists an infinite number of sets of the form $\{a, b, c, d\}$ with the property $D(\ell^2)$. We would like to describe the construction of those sets.

Let $ab+\ell^2=k^2$ and let s and t be positive integers satisfying the Pellian equation $S^2-abT^2=1$ (s and t exist since ab is not a perfect square). Two double sequences $y_{n,m}$ and $z_{n,m}$, $n,m\in Z$, can be defined as follows (see [5]):

$$y_{0,0}=\ell,\ z_{0,0}=\ell,\ y_{1,0}=k+a,\ z_{1,0}=k+b,$$
$$y_{-1,0}=k-a,\ z_{-1,0}=k-b,$$
$$y_{n+1,0}=\tfrac{2k}{\ell}y_{n,0}-y_{n-1,0},\ z_{n+1,0}=\tfrac{2k}{\ell}z_{n,0}-z_{n-1,0},\ n\in Z,$$
$$y_{n,1}=sy_{n,0}+atz_{n,0},\ z_{n,1}=bty_{n,0}+sz_{n,0},\ n\in Z,$$
$$y_{n,m+1}=2sy_{n,m}-y_{n,m-1},\ z_{n,m+1}=2sz_{n,m}-z_{n,m-1},\ n,m\in Z.$$

Let us write

$$x_{n,m}=(y_{n,m}^2-\ell^2)/a. \tag{1}$$

According to Theorem 2 of [5], if $x_{n,m}$ and $x_{n+1,m}$ are positive integers, then the set $\{a,b,x_{n,m},x_{n+1,m}\}$ has the property $D(\ell^2)$. It is also proved that the sets $\{a,b,x_{0,m},x_{1,m}\}$, $m\in Z\setminus\{-2,-1,0\}$, and $\{a,b,x_{-1,m},x_{0,m}\}$, $m\in Z\setminus\{-1,0,1\}$, have the property $D(\ell^2)$. So, it is sufficient to find one positive integer solution of the Pellian equation $S^2-abT^2=1$ to extend a set $\{a,b\}$ with the property $D(\ell^2)$ to a set $\{a,b,c,d\}$ with the same property.

2.2 Generalized Fibonacci Numbers

In [9], the generalized Fibonacci sequence of numbers (w_n) was defined by Horadam as follows: $w_n=w_n(a,b;p,q)$, $w_0=a$, $w_1=b$, $w_n=pw_{n-1}-qw_{n-2}$ $(n\geq 2)$, where $a,b,p,$ and q are integers. The properties of that sequence were discussed in detail in [10], [11], and [13]. The following identities have been proved:

$$w_nw_{n+2r}-eq^nU_r=w_{n+r}^2, \tag{2}$$

$$4w_nw_{n+1}^2w_{n+2}+(eq^n)^2=(w_nw_{n+2}+w_{n+1}^2)^2, \tag{3}$$

$$w_nw_{n+1}w_{n+3}w_{n+4}=w_{n+2}^4+eq^n(p^2+q)w_{n+2}^2+e^2q^{2n+1}p^2, \tag{4}$$

$$\begin{aligned}4w_nw_{n+1}w_{n+2}w_{n+4}w_{n+5}w_{n+6}&+e^2q^{2n}(w_nU_4U_5-w_{n+1}U_2U_6-w_nU_1U_8)^2\\&=(w_{n+1}w_{n+2}w_{n+6}+w_nw_{n+4}w_{n+5})^2.\end{aligned} \tag{5}$$

Here $e=pab-qa^2-b^2$ and $U_n=w_n(0,1;p,q)$. Identity (5) is due to Morgado [13].

Our purpose is to apply the above identities to constructing Diophantine quadruples. Considering the construction described in §2.1, we will restrict our attention to two special cases. For simplicity of notation, these are

$$u_n=u_n(p)=w_n(0,1;p,-1),\quad p\geq 1,$$
$$g_n=g_n(p)=w_n(0,1;p,1),\quad p\geq 2.$$

The Fibonacci sequence $F_n=u_n(1)$, the Pell sequence $P_n=u_n(2)$, the Fibonacci numbers of even subscript $F_{2n}=g_n(3)$, and $g_n(2)=n$ are important special cases of the above sequences.

Apart from the sequences (u_n) and (g_n), we also wish to investigate joined sequences (v_n) and (h_n), which are defined by $v_n=u_{n-1}+u_{n+1}$, $h_n=g_{n+1}-g_{n-1}$. It is easy to check that $v_n=w_n(2,p;p,-1)$ and $h_n=w_n(2,p;p,1)$.

3. QUADRUPLES WITH PROPERTIES $D(p^2u_n^2)$ AND $D(h_n^2)$

For every positive integer n,

$$4u_nu_{n+2}+(pu_{n+1})^2=v_{n+1}^2. \tag{6}$$

Indeed, $v_{n+1}^2-(pu_{n+1})^2=(u_n+u_{n+2})^2-(u_{n+2}-u_n)^2=4u_nu_{n+2}$. From the above, it follows that the sets $\{2u_n, 2u_{n+2}\}$, $\{u_n, 4u_{n+2}\}$, and $\{4u_n, u_{n+2}\}$ have the property $D(p^2u_{n+1}^2)$. In order to extend these sets to the quadruples with the property $D(p^2u_{n+1}^2)$ by applying the construction described in §2.1, it is necessary to find a solution to the Pellian equation $S^2-4u_nu_{n+2}T^2=1$. One solution of this equation can be obtained from the identity

$$4u_nu_{n+1}^2u_{n+2}+1=(u_{n+1}^2+u_nu_{n+2})^2, \tag{7}$$

which is the direct consequence of (2). Therefore, we will set $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. Now, applying the construction from §2.1, we obtain an infinite number of sets with the property $D(p^2u_{n+1}^2)$. In particular, we have

Theorem 1: Let n and p be positive integers. Then the six sets

$$\{2u_n, 2u_{n+2}, 2p^2u_{n+1}^3(u_{n+1}-u_n)(u_{n+2}-u_n), 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2})\},$$

$$\{2u_n, 2u_{n+2}, 2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2}),$$
$$2(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+2u_{n+1}+u_{n+2})(u_nu_{n+1}+2u_nu_{n+2}+u_{n+1}u_{n+2})\},$$

$$\{u_n, 4u_{n+2}, (u_{n+1}-u_n)(u_{n+2}-u_{n+1})(2u_{n+2}-u_n-u_{n+1})(2u_{n+1}u_{n+2}-u_nu_{n+1}-u_nu_{n+2}),$$
$$p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2})\},$$

$$\{u_n, 4u_{n+2}, p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2}),$$
$$(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+3u_{n+1}+2u_{n+2})(u_nu_{n+1}+3u_nu_{n+2}+2u_{n+1}u_{n+2})\},$$

$$\{4u_n, u_{n+2}, (u_{n+1}-u_n)(u_{n+1}+u_{n+2}-2u_n)(u_nu_{n+2}+u_{n+1}u_{n+2}-2u_nu_{n+1}),$$
$$p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2})\},$$

and

$$\{4u_n, u_{n+2}, p^2u_{n+1}^3(2u_n+u_{n+1})(2u_{n+1}+u_{n+2}),$$
$$(u_n+u_{n+1})(u_{n+1}+u_{n+2})(2u_n+3u_{n+1}+u_{n+2})(2u_nu_{n+1}+3u_nu_{n+2}+u_{n+1}u_{n+2})\}$$

have the property $D(p^2u_{n+1}^2)$.

Proof: The main idea of the proof is to show that the six sets in Theorem 1 are of the form $\{a, b, x_{0,1}, x_{1,1}\}$ or $\{a, b, x_{-1,1}, x_{0,1}\}$. Combining (6) with (7), we obtain $\ell=pu_{n+1}$, $k=v_{n+1}$, $s=u_{n+1}^2+u_nu_{n+2}$, $t=u_{n+1}$. To simplify notation, we write $u_{n+2}=A$, $u_{n+1}=B$. Hence, according to (2), $A^2-pAB-B^2=(-1)^{n+1}$, and that gives

$$(A^2-pAB-B^2)^2=1. \tag{8}$$

We arrange the proof in three parts, each part relating to two of the six sets.

Part 1. $\boldsymbol{a=2u_n}$, $\boldsymbol{b=2u_{n+2}}$

Using the notation of §2.1, we have

$$y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=3u_n+u_{n+2},\ z_{1,0}=u_n+3u_{n+2},$$
$$y_{-1,0}=pu_{n+1},\ z_{-1,0}=-pu_{n+1}.$$

From this, we obtain

$$y_{0,1}=pB[A^2+(2-p)AB-(2p-1)B^2],$$
$$y_{1,1}=4A^3+(8-7p)A^2B+(3p^2-10p+4)AB^2+p(2p-3)B^3,$$
$$y_{-1,1}=pB[A^2-(p+2)AB+(2p+1)B^2].$$

Relation (8) will be used to represent expressions of $x_{i,1}$, $i=-1,0,1$, obtained by putting $y_{i,1}$ in (1), as homogeneous polynomials in two variables A and B. When those polynomials are factored, we have

$$x_{0,1}=2p^2B^3\{A-(p-1)B](A+B)=2p^2u_{n+1}^3(u_n+u_{n+1})(u_{n+1}+u_{n+2}),$$
$$x_{1,1}=2[A-(p-1)B]A+B)[2A-(p-2)B][2A^2-2(p-1)AB-pB^2]$$
$$=2(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+2u_{n+1}+u_{n+2})(u_nu_{n+1}+2u_nu_{n+2}+u_{n+1}u_{n+2}),$$
$$x_{-1,1}=2p^2B^3[(p+1)B-A](A-B)=2p^2u_{n+1}^3(u_{n+1}-u_n)(u_{n+2}-u_{n+1}).$$

Part 2. $a=u_n,\ b=4u_{n+2}$

We now have

$$y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=2u_n+u_{n+2},\ z_{1,0}=u_n+5u_{n+2},$$
$$y_{-1,0}=u_{n+2},\ z_{-1,0}=u_n-3u_{n+2}.$$

Hence

$$y_{0,1}=pB[A^2-(p-1)AB-(p-1)B^2],$$
$$y_{1,1}=3A^3-(5p-6)A^2B+(2p^2-7p+3)AB^2+p(p-2)B^3,$$
$$y_{-1,1}=A^3-(p+2)A^2B+(p+1)AB^2+pB^3,$$

and, from (1) and (8),

$$x_{0,1}=p^2B^3(A+2B)[2A-(p-1)B]=p^2u_{n+1}^3(u_n+2u_{n+1})(u_{n+1}+2u_{n+2}),$$
$$x_{1,1}=[A-(p-1)B](A+B)[3A-(p-3)B][3A^2-3(p-1)AB-pB^2]$$
$$=(u_n+u_{n+1})(u_{n+1}+u_{n+2})(u_n+3u_{n+1}+2u_{n+2})(u_nu_{n+1}+3u_nu_{n+2}+2u_{n+1}u_{n+2}),$$
$$x_{-1,1}=[A-(p-1)B][A-(p+1)B](A-B)[A^2-(p+1)AB-pB^2]$$
$$=(2u_{n+2}-u_n-u_{n+1})(u_{n+1}-u_n)(u_{n+2}-u_{n+1})(2u_{n+1}u_{n+2}-u_nu_{n+1}-u_nu_{n+2}).$$

Part 3. $a=4u_n,\ b=u_{n+2}$

In this case,

$$y_{0,0}=z_{0,0}=pu_{n+1},\ y_{1,0}=5u_n+u_{n+2},\ z_{1,0}=u_n+2u_{n+2},$$
$$y_{-1,0}=u_{n+2}-3u_n,\ z_{-1,0}=u_n.$$

Accordingly,

$$y_{0,1} = pB[A^2-(p-4)AB-(4p-1)B^2],$$
$$y_{1,1} = 6A^3-(11p-12)A^2B+(5p^2-16p+6)AB^2+p(4p-5)B^3,$$
$$y_{-1,1} = -2A^3+(5p+4)A^2B-(3p^2+8p+2)AB^2+p(4p+3)B^3,$$

and, finally,

$$x_{0,1} = p^2B^3(A+2B)[2A-(2p-1)B] = p^2u_{n+1}^3(2u_{n+1}+u_{n+2})(2u_n+u_{n+1}),$$
$$x_{1,1} = [A-(p-1)B](A+B)[3A-(2p-3)B][3A^2-3(p-1)AB-2pB^2]$$
$$= (u_n+u_{n+1})(u_{n+1}+u_{n+2})(2u_n+3u_{n+1}+u_{n+2})(2u_nu_{n+1}+3u_nu_{n+2}+u_{n+1}u_{n+2}),$$
$$x_{-1,1} = [A-(p+1)B][A-(2p+1)B](A-B)[A^2-(p+1)AB+2pB^2]$$
$$= (u_{n+1}-u_n)(u_{n+2}-u_{n+1})(u_{n+1}+u_{n+2}-2u_n)(u_nu_{n+2}+u_{n+1}u_{n+2}-2u_nu_{n+1}). \quad \square$$

Using the identities $4g_ng_{n+2}+h_{n+1}^2 = p^2g_{n+1}^2$ and $4g_ng_{n+1}^2g_{n+2}+1=(g_{n+1}^2+g_ng_{n+2})^2$, we find the following theorem may be proved in much the same way as Theorem 1.

Theorem 2: Let $n \geq 1$ and $p \geq 2$ be integers. Then the six sets

$$\{2g_n,\, 2g_{n+2},\, 2g_{n+1}h_{n+1}^2(g_{n+1}-g_n)(g_{n+2}-g_{n+1}),\, 2g_{n+1}h_{n+1}^2(g_n+g_{n+1})(g_{n+1}+g_{n+2})\},$$

$$\{2g_n,\, 2g_{n+2},\, 2g_{n+1}h_{n+1}^2(g_n+g_{n+1})(g_{n+1}+g_{n+2}),$$
$$2(p+2)g_{n+1}(g_n+g_{n+1})(g_{n+1}+g_{n+2})(g_ng_{n+1}+2g_ng_{n+2}+g_{n+1}g_{n+2})\},$$

$$\{g_n,\, 4g_{n+1},\, (g_{n+1}-g_n)(g_{n+2}-g_{n+1})(2g_{n+2}-g_n-g_{n+1})(2g_{n+1}g_{n+2}-g_ng_{n+1}-g_ng_{n+2}),$$
$$g_{n+1}h_{n+1}^2(g_n+2g_{n+1})(g_{n+1}+2g_{n+2})\},$$

$$\{g_n,\, 4g_{n+2},\, g_{n+1}h_{n+1}^2(g_n+2g_{n+1})(g_{n+1}+g_{n+2}),$$
$$(g_n+g_{n+1})(g_{n+1}+g_{n+2})(g_n+3g_{n+1}+2g_{n+2})(g_ng_{n+1}+3g_ng_{n+2}+2g_{n+1}g_{n+2})\}$$

$$\{4g_n,\, g_{n+2},\, (g_{n+1}-g_n)(g_{n+2}-g_{n+1})(g_{n+1}+g_{n+2}-2g_n)(g_ng_{n+2}+g_{n+1}g_{n+2}-2g_ng_{n+1}),$$
$$g_{n+1}h_{n+1}^2(2g_n+g_{n+1}+g_{n+2})\},$$

and

$$\{4g_n,\, g_{n+2},\, g_{n+1}h_{n+1}^2(2g_n+g_{n+1})(2g_{n+1}+g_{n+2}),$$
$$(g_n+g_{n+1})(g_{n+1}+g_{n+2})(2g_n+3g_{n+1}+g_{n+2})(2g_ng_{n+1}+3g_ng_{n+2}+g_{n+1}g_{n+2})\}$$

have the property $D(h_{n+1}^2)$.

4. THE MORGADO IDENTITY

We are now going to use the Morgado identity (5). It is easy to check that

$$w_nU_4U_5 - w_{n+1}U_2U_6 - w_nU_1U_8 = U_2U_3(w_{n+4}-qw_{n+2}),$$
$$w_{n+1}w_{n+2}w_{n+6}+w_nw_{n+4}w_{n+5} = w_{n+3}(eq^nU_2^2U_3+2w_{n+2}w_{n+4}).$$

If we restrict the discussion to the sequences (u_n) and (g_n), the Morgado identity can be used as a base for constructing quadruples with the properties $D((u_2u_3v_{n+3})^2)$ and $D((g_2g_3h_{n+3})^2)$.

We are again going to use the construction described in §2.1. This time it is not necessary to use the solutions of the Pellian equation. We will try to choose the numbers a and b in the manner that the solution of the problem can be obtained using only the sequence $(x_{n,0})$. According to §2.1, if $x_{2,0} \in N$ or $x_{-2,0} \in N$, then, respectively, $\{a, b, x_{1,0}, x_{2,0}\}$ and $\{a, b, x_{-1,0}, x_{-2,0}\}$ are Diophantine quadruples.

Since $y_{2,0} = \frac{2k}{\ell}(k+a) - \ell$, $y_{-2,0} = \frac{2k}{\ell}(k-a) - \ell$, we have

$$x_{2,0} = \frac{y_{2,0}^2 - \ell^2}{a} = \frac{4k(k+a)(k+b)}{\ell^2} = \frac{4k}{\ell^2}(kx_{1,0} - \ell^2),$$

$$x_{-2,0} = \frac{y_{-2,0}^2 - \ell^2}{a} = \frac{-4k(k-a)(k-b)}{\ell^2} = \frac{4k}{\ell^2}(kx_{-1,0} + \ell^2).$$

Theorem 3: Let n and p be positive integers and $k = u_{n+3}[2u_{n+2}u_{n+4} - (-1)^n p^2(p^2+1)]$. Then the three sets

$$\left\{2u_n u_{n+1} u_{n+2},\ 2u_{n+4}u_{n+5}u_{n+6},\ 2(p^2+1)^2 u_{n+3}u_{n+3}^2,\ 4k\left(\frac{2ku_{n+3}}{p^2} - 1\right)\right\},$$

$$\left\{2u_n u_{n+1} u_{n+4},\ 2u_{n+2}u_{n+5}u_{n+6},\ 2p^2 u_{n+3}v_{n+3}^2,\ 4k\left(\frac{2ku_{n+3}}{(p^2+1)^2} + 1\right)\right\},$$

and

$$\left\{2u_n u_{n+2} u_{n+5},\ 2u_{n+1}u_{n+4}u_{n+6},\ 2u_{n+3}v_{n+3}^2,\ 4k\left(\frac{2ku_{n+3}}{p^2(p^2+1)^2} - 1\right)\right\}$$

have the property $D(p^2(p^2+1)^2 v_{n+3}^2)$.

Proof: The proof is by applying the construction from §2.1 to identity (5) for $w_n = u_n$. Three cases need to be considered.

Case 1. $\boldsymbol{a = 2u_n u_{n+1} u_{n+2},\ b = 2u_{n+4}u_{n+5}u_{n+6}}$

Hence, $a + b = 2(p^2+2)u_{n+3}[(p^2+1)(u_{n+2}^2 + u_{n+4}^2) + (p^2-1)u_{n+2}u_{n+4}]$. This gives

$$x_{1,0} = a + b + 2k = 2(p^2+1)^2 u_{n+3}(u_{n+2} + u_{n+4})^2 = 2(p^2+1)^2 u_{n+3}v_{n+3}^2,$$

$$x_{2,0} = 4k\left(\frac{k \cdot 2(p^2+1)^2 u_{n+3}v_{n+3}^2}{p^2(p^2+1)^2 v_{n+3}^2} - 1\right) = 4k\left(\frac{2ku_{n+3}}{p^2} - 1\right).$$

Case 2. $\boldsymbol{a = 2u_n u_{n+1} u_{n+4},\ b = 2u_{n+2}u_{n+5}u_{n+6}}$

Now we have $a + b = 2u_{n+3}[(p^2+1)(p^2+4)u_{n+2}u_{n+4} - u_{n+2}^2 - u_{n+4}^2]$ and

$$x_{-1,0} = a + b - 2k = 2p^2 u_{n+3}v_{n+3}^2,$$

$$x_{-2,0} = 4k\left(\frac{k \cdot 2p^2 u_{n+3}v_{n+3}^2}{p^2(p^2+1)^2 v_{n+3}^2} + 1\right) = 4k\left(\frac{2ku_{n+3}}{(p^2+1)^2} + 1\right).$$

Case 3. $\boldsymbol{a = 2u_n u_{n+2} u_{n+5},\ b = 2u_{n+1}u_{n+4}u_{n+6}}$

We have $a + b = 2(p^2+2)u_{n+3}[u_{n+2}^2 + u_{n+4}^2 - (p^2+1)u_{n+2}u_{n+4}]$ and

$$x_{1,0} = 2u_{n+3}v_{n+3}^2,$$
$$x_{2,0} = 4k\left(\frac{2ku_{n+3}}{p^2(p^2+1)^2} - 1\right).$$

It remains to prove that all elements of the sets from this theorem are integers. It is sufficient to prove that the number $8k^2u_{n+3}/p^2(p^2+1)^2$ is an integer for all positive integers n. That is the direct consequence of the relation

$$\frac{8k^2u_{n+3}}{p^2(p^2+1)^2} = \frac{8u_{n+3}^3[p^4(p^2+1)^2 - (-1)^n 4p^2(p^2+1)u_{n+2}u_{n+4} + 4u_{n+2}^2u_{n+4}^2]}{u_2^2u_3^2}$$

and the fact that $u_2 \mid u_{2m}$ and $u_3 \mid u_{3m}$ for all $m \in N$, which is easy to prove by induction. □

The following theorem can be proved in much the same way as Theorem 3.

Theorem 4: Let $n \geq 1$ and $p \geq 2$ be integers and $k = g_{n+3}[2g_{n+2}g_{n+4} - p^2(p^2-1)]$. Then the three sets

$$\left\{2g_ng_{n+1}g_{n+2}, 2g_{n+4}g_{n+5}g_{n+6}, 2(p^2-1)^2g_{n+3}h_{n+3}^2, 4k\left(\frac{2kg_{n+3}}{p^2}+1\right)\right\},$$

$$\left\{2g_ng_{n+1}g_{n+4}, 2g_{n+2}g_{n+5}g_{n+6}, 2p^2g_{n+3}h_{n+3}^2, 4k\left(\frac{2kg_{n+3}}{(p^2-1)^2}-1\right)\right\},$$

and

$$\left\{2g_ng_{n+2}g_{n+5}, 2g_{n+1}g_{n+4}g_{n+6}, 2g_{n+3}h_{n+3}^2, 4k\left(\frac{2kg_{n+3}}{p^2(p^2-1)^2}+1\right)\right\}$$

have the property $D(p^2(p^2-1)^2h_{n+3}^2)$.

We now want to show that the sequence (g_n) possesses another interesting property based on the identity

$$g_ng_{n+1}g_{n+3}g_{n+4} + [(p\pm 1)g_{n+2}]^2 = (g_{n+2}^2 \pm p)^2. \tag{9}$$

Now, the construction described in §2.1 can be applied on the relation (9). We have $a = g_ng_{n+1}$, $b = g_{n+3}g_{n+4}$, $k = g_{n+2}^2 \pm p$, which gives

$$x_{\mp 1,0} = a + b \mp 2k = (p^3 - 3p \mp 2)g_{n+2}^2 = (p\pm 1)^2(p\mp 2)g_{n+2}^2,$$
$$x_{\mp 2,0} = 4(g_{n+2}^2 \pm p)(g_{n+1} \mp g_n)(g_{n=4} \mp g_{n+3}).$$

Thus, we have proved

Theorem 5: Let $n \geq 1$ and $p \geq 2$ be integers. Then the set

$$\{g_ng_{n+1}, g_{n+3}g_{n+4}, (p+1)^2(p-2)g_{n+2}^2, 4(g_{n+2}^2+p)(g_{n+1}-g_n)(g_{n+4}-g_{n+3})\}$$

has the property $D((p+1)^2g_{n+2}^2)$, and the set

$$\{g_ng_{n+1}, g_{n+3}g_{n+4}, (p-1)^2(p+2)g_{n+2}^2, 4(g_{n+2}^2-p)(g_{n+1}+g_n)(g_{n+3}+g_{n+4})\}$$

has the property $D((p-1)^2g_{n+2}^2)$.

5. GENERALIZATION OF A RESULT OF BERGUM

Hoggatt and Bergum [8] have proved that the set

$$\{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\} \tag{10}$$

has the property $D(1)$ for every positive integer n. It has been proved in [4] that the set

$$\{F_{2n}, F_{2n+4}, 5F_{2n+2}, 4L_{2n+1}F_{2n+2}L_{2n+3}\} \tag{11}$$

also has the property $D(1)$. In [5], quadruples with the properties $D(4)$, $D(9)$, and $D(64)$ have been found using Fibonacci numbers. We now want to extend these results to the sequences (u_n) and (g_n) starting from identity (2). Applying (2) to the sequence (u_n), we get

$$u_{2n}\cdot u_{2n+2r}+u_r^2=u_{2n+r}^2. \tag{12}$$

Therefore, the sets $\{u_{2n}, u_{2n+2}\}$ and $\{u_{2n}, u_{2n+4}\}$ have, respectively, the properties $D(1)$ and $D(p^2)$ for every positive integer n. It was shown in §4 that, if a, b, k, and ℓ are the positive integers such that $ab+\ell^2=k^2$ and if the number $\pm 4k(k\pm a)(k\pm b)/\ell^2$ is a positive integer, then the set $\{a, b, a+b\pm 2k, \pm 4k(k\pm a)(k\pm b)/\ell^2\}$ has the property $D(\ell^2)$. According to this, we have

Theorem 6: Let n and p be positive integers. Then the sets

$$\{u_{2n}, u_{2n+2}, 2u_{2n}+(p-2)u_{2n+1}, 4u_{2n+1}[(p-2)u_{2n+1}^2+2u_{2n}u_{2n+1}+1]\}$$

and

$$\{u_{2n}, u_{2n+2}, 2u_{2n}-(p-2)u_{2n+1}, 4u_{2n+1}[2u_{2n+1}u_{2n+2}-(p-2)u_{2n+1}^2-1]\}$$

have the property $D(1)$ and the set

$$\{u_{2n}, u_{2n+4}, p^2u_{2n+2}, 4u_{2n+1}u_{2n+2}u_{2n+3}\}$$

has the property $D(p^2)$.

For the sequence (g_n), we can prove an even stronger result, namely, from (2) we have

$$g_n\cdot g_{n+2r}+g_r^2=g_{n+r}^2 \tag{13}$$

for every (not just even) positive integer n. Starting from the sets $\{g_n, g_{n+2}\}$ and $\{g_n, g_{n+4}\}$ with the properties $D(1)$ and $D(p^2)$, respectively, we find that the following result may be proved in much the same way as Theorem 6.

Theorem 7: Let $n\geq 1$ and $p\geq 2$ be integers. Then the sets

$$\{g_n, g_{n+2}, (p-2)g_{n+1}, 4g_{n+1}[(p-2)g_{n+1}^2+1]\}$$

and

$$\{g_n, g_{n+2}, (p+2)g_{n+1}, 4g_{n+1}[(p+2)g_{n+1}^2-1]\}$$

have the property $D(1)$, and the set

$$\{g_n, g_{n+4}, p^2g_{n+2}, 4g_{n+1}g_{n+2}g_{n+3}\}$$

has the property $D(9)$.

6. APPLICATION TO THE PELL NUMBERS AND POLYNOMIALS

In this section we apply the results discussed in the previous sections to some special cases of the sequences (u_n) and (g_n). The case of the Fibonacci sequence $F_n = u_n(1)$ and the case of the joined Lucas sequence $L_n = v_n(1)$ are studied in detail in [6].

Let us first examine the Pell sequence $P_n = u_n(2)$ and the Pell-Lucas sequence $Q'_n = v_n(2)$. All elements of the sequence (Q'_n) are even numbers, so we can write $Q'_n = 2Q_n$. The numbers P_n and Q_n are the solutions of the Pellian equation $x^2 - 2y^2 = \pm 1$. Namely, it is true that

$$Q_n^2 - 2P_n^2 = (-1)^n.$$

The sequences (P_n) and (Q_n) are related by relation $P_n + P_{n+1} = Q_{n+1}$. Applying this relation to Theorem 1, we get

Corollary 1: For every positive integer n, the sets

$$\{P_n, P_{n+2}, 4P_{n+1}^3 Q_n Q_{n+1}, 4P_{n+1}^3 Q_{n+1} Q_{n+2}\}$$

and

$$\{P_n, P_{n+2}, 4P_{n+1}^3 Q_{n+1} Q_{n+2}, 4P_{n+2} Q_{n+1} Q_{n+2}[P_{n+1}P_{n+2} - (-1)^n]\}$$

have the property $D(P_{n+1}^2)$.

In [6], quadruples with the property $D(L_{n+2}^2)$ are constructed using the following identities:

$$4F_n F_{n+4} + L_{n+2}^2 = 9F_{n+2}^2, \tag{14}$$

$$4F_n F_{n+2}^2 F_{n+4} + 1 = (F_{n+2}^2 + F_n F_{n+4})^2. \tag{15}$$

For the sequences (u_n), the following analogs of the above identities are valid:

$$4u_n u_{n+4} + (pv_{n+2})^2 = [(p^2+2)u_{n+2}]^2, \tag{16}$$

$$4u_{n+4} u_{n+2}^2 u_{n+4} + p^4 = (u_{n+2}^2 + u_n u_{n+4})^2. \tag{17}$$

Unfortunately, existence of the term p^4 in (17) makes it impossibie to apply the construction for finding quadruples with the property $D(p^2 v_{n+2}^2)$ from §2.1. But in the case $p = 2$, the solution of the equation $S^2 - abT^2 = 4$ can be obtained from relation (17). Thus, we can apply the modified construction described in Remark 1 of [5].

Theorem 8: For every positive integer n, the sets

$$\{P_n, P_{n+4}, 4P_{n+1}P_{n+2}P_{n+3}Q_{n+2}^2, 4P_{n+2}Q_{n+1}Q_{n+2}^2 Q_{n+3}\}$$

and

$$\{P_n, P_{n+4}, 4P_{n+2}Q_{n+1}Q_{n+2}^2 Q_{n+3}, 16P_{n+2}Q_{n+1}Q_{n+3}(2P_{n+2}^2 - P_{n+1}P_{n+3})\}$$

have the property $D(4Q_{n+2}^2)$.

Proof: The sets from Theorem 8 are easily seen to be of the forms $\{a, b, x'_{-1,1}, x'_{0,1}\}$ and $\{a, b, x'_{0,1}, x'_{1,1}\}$, respectively, where the sequence $(x'_{n,m})$ is constructed as described in Remark 1 of [5], that is, by setting $a = P_n$, $b = P_{n+4}$, $s' = P_{n+2}^2 + P_n P_{n+4}$, $t' = P_{n+2}$. □

In distinction from the identities (16) and (17), the construction from §2.1 can be applied directly to the following identities:

$$Q_nQ_{n+2}+Q_{n+1}^2=4P_{n+1}^2, \tag{18}$$

$$Q_nQ_{n+1}^2Q_{n+2}+1=4P_{n+1}^4. \tag{19}$$

We have thus proved

Theorem 9: For every positive integer n, the sets

$$\{Q_n, Q_{n+2}, 4P_nP_{n+1}Q_{n+1}^3, 4P_{n+1}P_{n+2}Q_{n+1}^3\}$$

and

$$\{Q_n, Q_{n+2}, 4P_{n+1}P_{n+2}Q_{n+1}^3, 4P_{n+1}P_{n+2}Q_{n+2}(P_{n+1}P_{n+3}-P_nP_{n+2})\}$$

have the property $D(Q_{n+1}^2)$.

Obviously, Theorems 3 and 6 can also be applied to the sequence (P_n). However, applying Theorem 6, as it is done for Fibonacci numbers in Theorem 3 of [5], gives us more.

Corollary 2: For every positive integer n, the sets

$$\{P_{2n}, P_{2n+2}, 2P_{2n}, 4P_{2n+1}Q_{2n}Q_{2n+1}\}$$

and

$$\{P_{2n}, P_{2n+2}, 2P_{2n+2}, 4P_{2n+1}Q_{2n+1}Q_{2n+2}\}$$

have the property $D(1)$, the sets

$$\{P_{2n}, P_{2n+4}, 4P_{2n+2}, 4P_{2n+1}P_{2n+2}P_{2n+3}\}$$

and

$$\{P_{2n}, P_{2n+4}, 8P_{2n+2}, 4P_{2n+2}Q_{2n+1}Q_{2n+3}\}$$

have the property $D(4)$, and the set

$$\{P_{2n}, P_{2n+8}, 36P_{2n+4}, P_{2n+2}P_{2n+4}P_{2n+6}\}$$

has the property $D(144)$.

In this paper only the quadruples with the property $D(n)$, where n is a perfect square, have been examined. However, let us mention that the set

$$\{1, P_{2n+1}(3P_{2n+1}-2), 3P_{2n+1}^2-1, P_{2n+1}(3P_{2n+1}+2)\}$$

has the property $D(-Q_{2n+1}^2)$ for every positive integer n.

Since $g_n(2)=n$, the results from this paper can be used to obtain the sets with the property of Diophantus whose elements are polynomials. For example, from Theorem 7, we get the Jones result that the set $\{n, n+2, 4(n+1), 4(n+1)(2n+1)(2n+3)\}$ has the property $D(1)$ for every positive integer n (see [12]).

The following interesting property of the binomial coefficients can be obtained as a consequence of the results from §4 above.

For every positive integer $n\geq 4$, the sets

$$\left\{\binom{n-1}{3}, \binom{n+3}{3}, 6n, \frac{2n(n^2-7)(n^2-3n+1)(n^2+3n-1)}{3}\right\}$$

and

$$\left\{\binom{n-1}{3}, \binom{n+3}{3}, \frac{2n(n^2+2)}{3}, \frac{2n(n^2-7)(n^3-3n^2+2n-3)(n^3+3n^2+2n+3)}{27}\right\}$$

have the property $D(1)$. Note that $h_n(2)=2$.

Finally, let us mention that, using these results, the explicit formulas for quadruples with the property $D(\ell^2)$, for a given integer ℓ, can be obtained. Of course, only the sets with at least one element that is not divisible by ℓ are of any interest to us here.

Corollary 3: Let ℓ be an integer. The sets

$$\{(\ell-1)(\ell-2), (\ell+1)(\ell+2), 4\ell^2, 2(2\ell-3)(2\ell+3)(\ell^2-2)\}, \text{ for } \ell \geq 3, \tag{20}$$

and

$$\{1, \ell^4-3\ell^2, \ell^2(\ell^2-1), 4\ell^2(\ell^2-1)(\ell^2-2)\}, \text{ for } \ell \geq 2, \tag{21}$$

have the property $D(\ell^2)$.

Proof: We can get set (20) by putting $p=2$ and $n+2=\ell$ in the second set of Theorem 5. Since $g_1(p)=1$, $g_3(p)=p^2-1$, $g_5(p)=p^4-3p^2+1$, set (21) can be obtained by putting $n=1$ and $p=\ell$ in the third set of Theorem 7. □

Remark 1: One question still unanswered is whether any of the Diophantine quadruples from this paper can be extended to the Diophantine quintuple with the same property. In this connection, let us mention that it is proved in [7] that, for every integer ℓ and every set $\{a,b,c,d\}$ with the property $D(\ell^2)$, where $abcd \neq \ell^4$, there exists a rational number r, $r \neq 0$, such that the set $\{a,b,c,d,r\}$ has the property that the product of any two of its elements increased by ℓ^2 is a square of a rational number.

For example, if the method from [7] is applied to the second set in Corollary 3, we get

$$r = \frac{8\ell(\ell-1)(\ell+1)(\ell^2-2)(2\ell^2-3)(2\ell^4-4\ell^2+1)(2\ell^4-6\ell^2+3)}{[4(\ell-1)^2(\ell+1)^2(\ell^2-2)(\ell^2-\ell-1)(\ell^2+\ell-1)-1]^2}.$$

From this, for $\ell=2$, we have the set $\{89760, 128881, 644405, 1546572, 12372576\}$ with the property $D(4\cdot 359^4)$.

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AMS Classification Numbers: 11B37, 11B39, 11D09

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