ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Proposers should inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-808 ***Proposed by Paul S. Bruckman, Jalmiya, Kuwait***

Years after Mr. Feta's demise at Bellevue Sanitarium, a chance inspection of his personal effects led to the discovery of the following note, scribbled in the margin of a well-worn copy of Professor E. P. Umbugio's "22 / 7 Calculated to One Million Decimal Places":

> To divide "n-choose-one" into two other non-trivial "choose one's", "n-choose-two", or in general, "n-choose-m" into two non-trivial "choose-m's", for any natural m is always possible, and I have assuredly found for this a truly wonderful proof, but the margin is too narrow to contain it.

Because of the importance of this result, it has come to be known as Mr. Feta's Lost Theorem. We may restate it in the following form:

Solve the Diophantine equation $x^{\underline{m}} + y^{\underline{m}} = z^{\underline{m}}$ for $m \le x \le y \le z$, $m = 1, 2, 3, \ldots$, where $X^{\underline{m}} = X(X-1)(X-2)\cdots(X-m+1)$. Was Mr. Feta crazy?

B-809 ***Proposed by Pentti Haukkanen, University of Tampere, Finland***

Let k be a positive integer. Find a recurrence consisting of positive integers such that each positive integer occurs exactly k times.

B-810 ***Proposed by Herta Freitag, Roanoke, VA***

Let $\langle H_n \rangle$ be a generalized Fibonacci sequence defined by $H_{n+2} = H_{n+1} + H_n$ for $n > 0$ with initial conditions $H_1 = a$ and $H_2 = b$, where a and b are integers. Let k be a positive integer. Show that

$$\begin{vmatrix} H_n & H_{n+1} \\ H_{n+k+1} & H_{n+k+2} \end{vmatrix}$$

is always divisible by a Fibonacci number.

B-811 ***Proposed by Russell Euler, Maryville, MO***

Let n be a positive integer. Show that:

(a) if $n \equiv 0 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_2 + 1$;
(b) if $n \equiv 1 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_3 + 1$;
(c) if $n \equiv 2 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_2 - 1$;
(d) if $n \equiv 3 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_3 - 1$.

B-812 ***Proposed by John C. Turner, University of Waikato, Hamilton, New Zealand***

Let P, Q, and R be three points in space with coordinates $(F_{n-1}, 0, 0)$, $(0, F_n, 0)$, and $(0, 0, F_{n+1})$, respectively. Prove that twice the area of ΔPQR is an integer.

B-813 ***Proposed by Peter Jeuck, Mahwah, NJ***

Let $\langle X_n \rangle$, $\langle Y_n \rangle$, and $\langle Z_n \rangle$ be three sequences that each satisfy the recurrence $W_n = pW_{n-1} + qW_{n-2}$ for $n > 1$, where p and q are fixed integers. (The initial conditions need not be the same for the three sequences.) Let a, b, and c be any three positive integers. Prove that

$$\begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = 0.$$

SOLUTIONS

A Floored Sum

B-781 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 33, no. 1, February 1995)

Let $F(j) = F_j$. Find a closed form for $\sum_{k=0}^{n} F\left(k - \lfloor \sqrt{k} \rfloor^2\right)$.

Solution by Graham Lord, Princeton, NJ

If we write $k = m^2 + \ell$, where $\ell = 0, 1, 2, \ldots, 2m$, then $k - (\lfloor \sqrt{k} \rfloor)^2 = \ell$. This follows from the fact that $m^2 \le k < (m+1)^2$ or $m \le \sqrt{k} < m+1$, so that $m = \lfloor \sqrt{k} \rfloor$.

Hence, if $n = N^2 + P$, where $0 \le P \le 2N$, then

$$\begin{aligned}\sum_{k=0}^{n} F\left(k - \lfloor \sqrt{k} \rfloor^2\right) &= F_0 + (F_0 + F_1 + F_2) + (F_0 + F_1 + F_2 + F_3 + F_4) + \cdots \\ &\quad + (F_0 + F_1 + \cdots + F_{2N-2}) + (F_0 + F_1 + \cdots + F_P) \\ &= (F_2 - 1) + (F_4 - 1) + (F_6 - 1) + \cdots + (F_{2N} - 1) + (F_{P+2} - 1) \\ &= (F_0 + F_1) + (F_2 + F_3) + (F_4 + F_5) + \cdots + (F_{2N-2} + F_{2N-1}) + F_{P+2} - N\end{aligned}$$

$$= F_{2N+1} + F_{P+2} - N - 2$$
$$= F(2\lfloor\sqrt{n}\rfloor + 1) + F(n - \lfloor\sqrt{n}\rfloor^2 + 2) - \lfloor\sqrt{n}\rfloor - 2.$$

Above, we have made repeated use of the identity $F_1 + F_2 + \cdots + F_i = F_{i+2} - 1$, which is Identity (I_1) from [1].

Reference

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

The proposer generalized to Lucas numbers. He showed that, if $L(j) = L_j$, then

$$\Sigma_{k=0}^{n} L(k - \lfloor\sqrt{k}\rfloor^2) = L(2\lfloor\sqrt{n}\rfloor + 1) + L(n - \lfloor\sqrt{n}\rfloor^2 + 2) - \lfloor\sqrt{n}\rfloor - 2.$$

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Carl Libis, Igor O. Popov, David Zeitlin, and the proposer.

Sum of Three Squares

B-782 ***Proposed by László Cseh, Stuttgart, Germany, & Imre Merény, Budapest, Hungary (Vol. 33, no. 1, February 1995)***

Express $(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$ as the sum of three squares.

Solution by H.-J. Seiffert, Berlin, Germany

An easy calculation shows that, for all numbers a_1, a_2, a_3, b_1, b_2, and b_3,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) = (a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2.$$

(This is known as Lagrange's Identity. –Ed.) Now take $a_1 = F_{n+h}$, $a_2 = F_n$, $a_3 = F_h$, $b_1 = F_{n+k}$, $b_2 = -F_{n+h+k}$, and $b_3 = (-1)^{n+1}F_k$. Since (see [1], formula 3.32, or [2], formula 20a)

$$F_{n+h}F_{n+k} - F_nF_{n+h+k} = (-1)^n F_hF_k,$$

we have $a_1b_1 + a_2b_2 + a_3b_3 = 0$. Thus, by Lagrange's Identity, we find that

$$(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$$
$$= (F_nF_{n+k} + F_{n+h}F_{n+h+k})^2 + (F_hF_{n+h+k} - (-1)^n F_nF_k)^2 + (F_hF_{n+k} + (-1)^n F_{n+h}F_k)^2.$$

References

1. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester, England: Ellis Horwood Ltd., 1989.

Zeitlin found the general identity

$$(F_{n+h}^2 + H_n^2 + H_h^2)(F_{n+h+k}^2 + H_{n+k}^2 + F_k^2)$$
$$= (H_nH_{n+k} + F_{n+h}F_{n+h+k})^2 + (H_hF_{n+h+k} - (-1)^n H_nF_k)^2 + (H_hF_{n+k} + (-1)^n F_{n+h}F_k)^2$$

where $\langle H_n \rangle$ is any sequence satisfying $H_{n+2} = H_{n+1} + H_n$. No solver gave a general procedure for writing a Fibonacci expression as a sum of squares.

Also solved by Paul S. Bruckman, C. Georghiou, Russell Jay Hendel, Norbert Jensen, David Zeitlin, and the proposers.

Crazed Rational Functions

B-783 ***Proposed by David Zeitlin, Minneapolis, MN***
(Vol. 33, no. 1, February 1995)

Find a rational function $P(x, y)$ such that

$$P(F_n, F_{2n}) = \frac{105n^5 - 1365n^3 + 1764n}{25n^6 + 175n^4 - 5600n^2 + 5904}$$

for $n = 0, 1, 2, 3, 4, 5, 6$.

Solution by C. Georghiou, University of Patras, Greece

The values taken by the given expression when $n = 0, 1, 2, \ldots, 6$ are, respectively, 0, 1, 1/3, 1/2, 3/7, 5/11, and 4/9; or equivalently: 0, 1/1, 1/3, 2/4, 3/7, 5/11, and 8/18; i.e., F_n / L_n. Since $F_{2n} = F_n L_n$, it follows that $P(x, y) = x^2 / y$.

Several solvers pointed out that, in this solution, we have to handwave the case when $n = 0$ because x^2 / y is not defined at $(0, 0)$. The editor was hoping that some solver would find a function such as

$$P(x, y) = \frac{y^2 + 6054y + 5850x}{30079y - 35364x^2 + 4026x + 13164},$$

but no solver came up with such a crazed function.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Joseph J. Koštál, Bob Prielipp, H.-J. Seiffert, and the proposer.

Lucas in Disguise

B-784 ***Proposed by Herta Freitag, Roanoke, VA***
(Vol. 33, no. 2, May 1995)

Show that, for all n, $\alpha^{n-1}\sqrt{5} - L_{n-1} / \alpha$ is a Lucas number.

Solution by Thomas Leong, Staten Island, NY

Since $\beta = -1/\alpha$ and $\sqrt{5} - 1/\alpha = \sqrt{5} + \beta = \alpha$, we have

$$\begin{aligned} \alpha^{n-1}\sqrt{5} - L_{n-1} / \alpha &= \alpha^{n-1}\sqrt{5} - (\alpha^{n-1} + \beta^{n-1}) / \alpha \\ &= \alpha^{n-1}(\sqrt{5} - 1/\alpha) - \beta^{n-1} / \alpha \\ &= \alpha^n + \beta^n = L_n. \end{aligned}$$

Haukkanen found the formulas $\alpha^{n-1} - F_{n-1} / \alpha = F_n$, $\beta^{n-1} - F_{n-1} / \beta = F_n$, $-\beta^{n-1}\sqrt{5} - L_{n-1} / \beta = L_n$. Redmond generalized to an arbitrary second-order linear recurrence.

Also solved by Michel Ballieu, Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Andrej Dujella, Russell Euler, Peter Gilbert, Pentti Haukkanen, Norbert Jensen, Joseph J. Koštál, Can. A. Minh, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, Lawrence Somer, M. N. S. Swamy, and the proposer.

It's a Multiple of $a_n a_{n+1}$

B-785 *Proposed by Jane E. Friedman, University of San Diego, CA (Vol. 33, no. 2, May 1995)*

Let $a_0 = a_1 = 1$ and let $a_n = 5a_{n-1} - a_{n-2}$ for $n \geq 2$. Prove that $a_{n+1}^2 + a_n^2 + 3$ is a multiple of $a_n a_{n+1}$ for all $n \geq 1$.

Solution by Andrej Dujella, University of Zagreb, Croatia

We will prove by induction that

$$a_{n+1}^2 + a_n^2 + 3 = 5a_n a_{n+1} \quad \text{for all } n \geq 1.$$

For $n = 1$, we have: $16+1+3 = 5\cdot 1\cdot 4$. Let us suppose that the assumption holds for some positive integer n. Then,

$$\begin{aligned} a_{n+2}^2 + a_{n+1}^2 + 3 &= (5a_{n+1} - a_n)^2 + a_{n+1}^2 + 3 \\ &= 25a_{n+1}^2 - 10a_n a_{n+1} + a_n^2 + a_{n+1}^2 + 3 \\ &= 25a_{n+1}^2 - 10a_n a_{n+1} + 5a_n a_{n+1} \\ &= 5a_{n+1}(5a_{n+1} - a_n) \\ &= 5a_{n+1}a_{n+2}. \end{aligned}$$

Gilbert found that if $a_n = ka_{n-1} - a_{n-2}$, *then* $a_{n+1}^2 + a_n^2 + [ka_0a_1 - a_0^2 - a_1^2] = ka_n a_{n+1}$. *Redmond found that if* $a_n = ka_{n-1} - ra_{n-2}$, *then* $a_{n+1}^2 + ra_n^2 + [ka_0a_1 - ra_0^2 - a_1^2]r^n = ka_n a_{n+1}$.

Also solved by Brian D. Beasley, Paul S. Bruckman, Charles K. Cook, Russell Euler, Herta T. Freitag, Peter Gilbert, Norbert Jensen, Thomas Leong, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, Lawrence Somer, M. N. S. Swamy, and the proposer.

Finding Coefficients of an Identity

B-786 *Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. of Sci., Gujarat State, India (Vol. 33, no. 2, May 1995)*

If $F_{n+2k}^2 = aF_{n+2}^2 + bF_n^2 + c(-1)^n$, where a, b, and c depend only on k but not on n, find a, b, and c.

Solution 1 by Paul S. Bruckman, Jalmiya, Kuwait

Given that the indicated relation must be valid for all n, set $n = -2$, -1, and 0, respectively. This yields the following three equations:

$$\begin{aligned} b + c &= F_{2k-1}^2; \\ a + b - c &= F_{2k-1}^2; \\ a + c &= F_{2k}^2. \end{aligned}$$

Solving for the three unknowns a, b, and c and simplifying yields

$$a = F_{4k}/3, \quad b = -F_{4k-4}/3, \quad \text{and} \quad c = 2F_{2k}F_{2k-2}/3.$$

It is a trite but straightforward exercise to verify that these values do indeed make the given equation an identity, as claimed.

Solution 2 by Stanley Rabinowitz, Mathpro Press, Westford, MA

All problems of this nature can be solved by the following method. We want to find when the expression $E = F_{n+2k}^2 - aF_{n+2}^2 - bF_n^2 - c(-1)^n$ is identically 0 in the variable n. First, apply the reduction formula $F_{n+x} = (F_nL_x + L_nF_x)/2$ to isolate n in subscripts. Then apply the formula $F_n^2 = (L_n^2 - 4(-1)^n)/5$ to remove any powers of F_n. The result is

$$20E = [36a + 16b - 20c - 4L_{2k}^2](-1)^n + [-30a + 10F_{2k}L_{2k}]F_nL_n + [-14a - 4b + 5F_{2k}^2 + L_{2k}^2]L_n^2.$$

This is known as the canonical form of the expression (considering n a variable and k a constant). There is a theorem that says that a polynomial expression in F_n and L_n is identically 0 if and only if its canonical form is 0. (For more details, see my paper "Algorithmic Manipulation of Fibonacci Identities" in *Proceedings of the Sixth International Conference on Fibonacci Numbers and Their Applications.*) Thus, E will be identically 0 if and only if each of the above coefficients in square brackets is 0. That is, if and only if

$$\begin{aligned} 36a + 16b - 20c - 4L_{2k}^2 &= 0, \\ -30a + 10F_{2k}L_{2k} &= 0, \\ -14a - 4b + 5F_{2k}^2 + L_{2k}^2 &= 0. \end{aligned}$$

Solving these equations simultaneously for the unknowns a, b, and c in terms of the constant k shows that E is 0 (identically in n) if and only if

$$a = \frac{F_{2k}L_{2k}}{3}, \quad b = \frac{5F_{2k}^2}{4} - \frac{7F_{2k}L_{2k}}{6} + \frac{L_{2k}^2}{4}, \quad \text{and} \quad c = F_{2k}^2 - \frac{F_{2k}L_{2k}}{3}.$$

Also solved by Brian D. Beasley, Andrej Dujella, Peter Gilbert, Norbert Jensen, Can. A. Minh, Don Redmond, H.-J. Seiffert, Tony Shannon, Sahib Singh, and the proposer.

Addenda: Igor O. Popov was inadvertently omitted as a solver of Problems B-779 and B-780. C. Georghiou was inadvertently omitted as a solver of Problems B-760, B-761, B-763, and B-765.

Dr. Dresel has informed us of the passing away of Steven Vajda, whose book, *Fibonacci & Lucas Numbers, and the Golden Section*, is often quoted in this column. Dr. Vajda was born in Budapest on August 20, 1901, and died in Sussex on December 10, 1995. An obituary can be found in *The Times* (London, January 3, 1996).

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