ON THE LEAST SIGNIFICANT DIGIT OF ZECKENDORF EXPANSIONS

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1. INTRODUCTION

A well-known digital expansion is the so-called Zeckendorf number system [7], where every positive integer n can be written as

$$n=\sum_{k=0}^{L}\varepsilon_k F_k, \tag{1.1}$$

where F_k denotes the sequence of Fibonacci numbers given by $F_{k+2}=F_{k+1}+F_k$, $F_0=1$, and $F_1=2$ (cf. [5]). The digits ε_k are 0 or 1, and $\varepsilon_k\varepsilon_{k+1}=0$. Using the same recurrence relation but the initial values $L_0=3$ and $L_1=4$, the sequence L_k of Lucas numbers is defined. In a recent volume of *The Fibonacci Quarterly*, P. Filipponi proposed the following conjectures (Advanced Problem H-457, cf. [2]).

Conjecture 1: Let $f(N)$ denote the number of 1's in the Zeckendorf decomposition of N. For given positive integers k and n, there exists a minimal positive integer $R(k)$ (depending on k) such that $f(kF_n)$ has a constant value for $n\geq R(k)$.

Conjecture 2: For $k\geq 6$, let us define

(i) μ, the subscript of the smallest odd-subscripted Lucas number such that $k\leq L_\mu$,
(ii) ν, the subscript of the largest Fibonacci number such that $k>F_\nu+F_{\nu-6}$.

Then $R(k)=\max(\mu,\nu)+2$.

We note that we have chosen different initial values compared to [5] and [2] (the so-called "canonical" initial values, cf. [4]) which seem to be more suitable for defining digital expansions and yield an index translation by 2. In [3] we have proved that the first conjecture is true in a much more general situation, i.e., for digital expansions with respect to linear recurrences with nonincreasing coefficients. As in [3], let $U(k)$ be the smallest index u such that

$$kF_u=\sum_{\ell=0}^{L(k)}\varepsilon_\ell F_\ell \quad\text{and}\quad kF_n=\sum_{\ell=0}^{L(k)}\varepsilon_\ell F_{\ell+n-u}\ \forall n\geq u. \tag{1.2}$$

We prove an explicit formula for $U(k)$ in terms of Lucas numbers that is an improved version of Conjecture 1. Note that Filipponi's Conjecture 1 has been proved by Bruckman in [1] and for the more general case of digital expansions with respect to linear recurrences in [3]. We have also obtained a weak formulation of Conjecture 2 which only yields an upper bound for $U(k)$. However, Bruckman's proof of a modification of Filipponi's Conjecture 2 is false because his proof does not guarantee the minimality of $R(k)$; this was pointed out in a personal communication by Piero Filipponi. We apologize here for referring in [3] to this erroneous proof instead of presenting our own proof of the original Conjecture 2. It is the aim of this note to provide a complete proof of Conjecture 2.

2. PROOF OF CONJECTURE 2

In the following, let $V(k)=L(k)-U(k)$ be the largest power of the golden ratio $\beta=\frac{1+\sqrt{5}}{2}$ in Parry's β-expansion of k, see [6]. Obviously, $V(k)=\lfloor \log_\beta k \rfloor$. For proving Conjecture 2, let us intro-duce some special notation. By Zeckendorf's theorem, every nonnegative integer n can be written uniquely as

$$n=F_{k_r}+\cdots+F_{k_2}+F_{k_1},\; k_r \gg \cdots \gg k_2 \gg k_1,\; r\geq 0, \tag{2.2}$$

where $k' \gg k''$ means that $k' \geq k''+2$ [compare to (1.1)].

It will be convenient to have the sequences of Fibonacci and Lucas numbers extended for negative indices. Let $F_{-2}=0, F_{-1}=1, F_{-n-2}=(-1)^{n+1}F_{n-2}$ and $L_{-2}=2, L_{-1}=1, L_{-n-2}=(-1)^n L_{n-2}$ for positive integers n. In this way, the definitions of μ and ν hold for all integers. We need the following well-known lemmas which can be shown by induction.

Lemma 1: For integers m and n, we have $L_m F_n=(-1)^m F_{n-m}+F_{n+m}$.

Lemma 2: Let m and n be integers, $n>m$ and $m\equiv n \bmod 2$. Then

$$F_n-F_m=\sum_{i=1}^{\frac{n-m}{2}} F_{m+2i-1}.$$

Theorem 1: For all positive integers k there exist uniquely determined integers $c_1,\ldots,c_t$ such that, for all integers n,

$$kF_n=\sum_{i=1}^{t} F_{n+c_i} \tag{2.3}$$

with

$$-U(k)=c_1 \ll c_2 \ll \cdots \ll c_{t-1} \ll c_t=V(k), \tag{2.4}$$

where $U(k)\geq 2$ are even numbers defined by $L_{U(k)-3}<k\leq L_{U(k)-1}$.

Proof: We consider the following partition of the set of natural numbers $\mathbb{N}=\bigcup_{j=-1}^{\infty}\mathbb{L}_j$, where $\mathbb{L}_{-1}=\{1\}$ and $\mathbb{L}_j=\{n\in\mathbb{N} \mid L_{2j-1}<n\leq L_{2j+1}\}$ for $j\geq 0$. The proof will proceed by induction on j.

If $j=-1$, i.e., $k=1$, then the assertion is satisfied with $t=1$ and $c_1=0$. Suppose that (2.3) and (2.4) hold for $j\geq 0$ for each i with $-1\leq i\leq j-1$ and all $k\in\mathbb{L}_i$. Then we have to show (2.3) and (2.4) hold for all $k\in\mathbb{L}_j$. Three cases will be distinguished.

Case 1: $L_{2j-1}<k<L_{2j}$

From Lemma 2 with $m=2j+1$ and by $-F_{n-2j+1}=F_{n-2j}-F_{n-2j+2}$, we have

$$kF_n = F_{n-2j} - F_{n-2j+2} + (k - L_{2j-1})F_n + F_{n+2j-1}. \tag{2.5}$$

Since $1 \le k - L_{2j-1} < L_{2j-2}$, by the induction hypothesis we obtain from (2.5),

$$kF_n = F_{n-2j} - F_{n-2j+2} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1} \tag{2.6}$$

with $\bar{c}_1 \ge -2(j-1)$, $\bar{c}_{\bar{t}} \le 2(j-1)-1$, and $\bar{c}_1 \ll \cdots \ll \bar{c}_{\bar{t}}$. Write (2.6) in the form

$$kF_n = F_{n-2j} + F_{n+\bar{c}_1} - F_{n-2j+2} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1}. \tag{2.7}$$

If $\bar{c}_1 = -2j+2$, then by $\bar{c}_1 \ll \bar{c}_2$ we have $-2j+4 \le \bar{c}_2$. Letting $t = \bar{t}+1$, $c_1 = -2j$, and $c_2 = \bar{c}_2$, ..., $c_{t-1} = \bar{c}_{\bar{t}}$, then $c_1 \le c_2 - 4$. Thus, $c_1 \ll c_2$ and, by the induction hypothesis, $c_2 \ll \cdots \ll c_{t-1}$. If $\bar{c}_1 > -2(j-1)$, then Lemma 2 applies for $F_{n+\bar{c}_1} - F_{n-2j+2}$ since, by the induction hypothesis, $\bar{c}_1$ is a value of the even-valued function U. Hence, we get

$$kF_n = F_{n-2j} + \sum_{\ell=1}^{\hat{t}} F_{n-2j+2\ell+1} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j-1} \tag{2.8}$$

with $\hat{t} = (\bar{c}_i - 2(j-1))/2$. Representation (2.8) is already in the form (2.3). Letting $t = \bar{t} + \hat{t} + 2$ and $c_1 = -2j$, $c_2 = -2j+3, \ldots, c_{\hat{t}+1} = \bar{c}_1 - 1, c_{\hat{t}+2} = \bar{c}_2, \ldots, c_{\hat{t}+\bar{t}+1} = \bar{c}_{\bar{t}}$, and using $c_2 = c_1 + 3$, $c_{i+1} \ge c_i + 2$ $(i = 2, \ldots, \hat{t})$, we get $c_1 \ll c_2 \ll \cdots \ll c_{\hat{t}+1}$. Applying the induction hypothesis yields $c_{\hat{t}+2} \ll \cdots \ll c_{t-1}$. Taking $c_t = 2j-1$, (2.4) is established.

Case 2: $L_{2j} < k < L_{2j+1}$

From Lemma 1 with $m = 2j$ we derive

$$kF_n = F_{n-2j} + (k - L_{2j})F_n + F_{n+2j}. \tag{2.9}$$

Since $1 \le k - L_{2j} < L_{2j-1}$, the induction hypothesis yields a representation of the form (2.3),

$$kF_n = F_{n-2j} + \sum_{i=1}^{\bar{t}} F_{n+\bar{c}_i} + F_{n+2j}, \tag{2.10}$$

with $\bar{c}_1 \ge -2(j-1), \bar{c}_{\bar{t}} \ge 2(j-1)$, and $\bar{c}_1 \ll \cdots \ll \bar{c}_t$. Letting $t = \bar{t}+2, c_1 = -2j, c_t = 2j$, and $c_{i+1} = \bar{c}_i$ $(i = 1, \ldots, \hat{t})$, we obtain (2.4).

Case 3: $k = L_{2j}$

By Lemma 2 we have $L_{2j}F_n = F_{n-2j} + F_{n+2j}$. Thus, we can proceed without using the induction hypothesis, obtaining (2.3) and (2.4) with $t = 2, c_1 = -2j$, and $c_2 = 2j$.

Uniqueness of $c_1, \ldots, c_t$ is implied by the uniqueness of the Zeckendorf representation. □

Corollary 1: As an immediate consequence of Theorem 1, we get $R(k) \le U(k)$.

To prove Conjecture 2, we need an additional lemma.

Lemma 3: Let c_1 and c_2 be as in Theorem 1. Then $c_2 = c_1 + 2$ if and only if

$$k > 2L_{-c_1-1}. \tag{2.11}$$

Proof: By Theorem 1, we have $4F_n = F_{n-2} + F_n + F_{n+2}$; thus, $c_2 = c_1 + 2$. Also by Theorem 1, for $k \geq 5$, we obtain $c_2 \geq c_1 + 2$ and $c_1 = -2j$ for some integer $j \geq 1$. From the proof of that theorem, it is clear that $L_{2j-1} < k \geq L_{2j+1}$. If $L_{2j-1} < k < L_{2j}$, then $c_2 > c_1 + 2$. If $k = L_{2j}$, then $t = 2$ and $c_2 - c_1 = 4j > 2$. If $L_{2j} < k \leq L_{2j+1}$, then $0 < k - L_{2j} < L_{2j-1}$. Observing that (2.11) is equivalent to $k - L_{2j} > L_{2j-3}$, Theorem 1 yields $U(k - L_{2j}) > -2(j-1)$ if $0 < k - L_{2j} \leq L_{2(j-1)-1}$ and $U(k - L_{2j}) = -2(j-1)$ if $L_{2(j-1)-1} < k - L_{2j} \leq L_{2(j-1)+1}$. Thus, we conclude that $c_2 = -2j + 2$ if and only if (2.11) holds. □

Theorem 2: $R(1) = 0$, $R(2) = R(3) = 1$, and for $k \geq 4$ we have

$$R(k) = \begin{cases} 2j - 1 & \text{if } L_{2j-3} < k \leq 2L_{2j-3}, \\ 2j & \text{if } 2L_{2j-3} < k \leq L_{2j-1}. \end{cases}$$

Proof: $R(1) = 0$ is immediate from the definitions. By the identities $2F_n = F_{n-2} + F_{n+1}$, $3F_n = F_{n-2} + F_{n+2}$ for integral n, and $2F_1 = F_0 + F_2$, $3F_1 = F_0 + F_3$ we obtain $R(2) \geq 1$ and $R(3) \geq 1$. Since $2F_0 = F_1$ and $3F_0 = F_2$, we get $R(2) = R(3) = 1$. Let $k \geq 4$. By Corollary 1, we have $R(k) \leq U(k)$ and $f(kF_n) = t$ for $n \geq U(k)$.

In the following, we distinguish two cases.

Case 1: $2L_{2j-3} < k \leq L_{2j-1}$

Let $n = U(k) - 1$. We show that in this case $f(kF_n) < t$; hence, $R(k) = U(k)$. Theorem 1 and Lemma 3 yield

$$kF_n = F_{-1} + F_1 + \sum_{i=3}^{t} F_{n+c_i} = F_2 + \sum_{i=3}^{t} F_{n+c_i}. \tag{2.12}$$

If $n + c_3 > 3$, then the right-hand side of (2.12) is a Zeckendorf representation and $f(kF_n) = t - 1$. If $n + c_3 = 3$, then let i_0 be the largest $i \geq 2$ such that $c_i = c_{i-2} + 2$; let $i_0 = 1$ if such i does not exist. Then the right-hand side of (2.12) can be written in the form of a Zeckendorf representation as

$$F_{n+c_{i_0}+1} + \sum_{i>i_0}^{t} F_{n+c_i}. \tag{2.13}$$

Thus, $f(kF_n) = t - i_0 + 1$.

Case 2: $L_{2j-3} < k \leq 2L_{2j-3}$

We show $f(kF_n) = t$ provided that $n = U(k) - 1$; however, $f(kF_n) = t - 1$ for $n = U(k) - 2$. Hence, we have $R(k) = U(k) - 1$. Let $n = U(k) - 1$. As a consequence of Theorem 1, we get

$$kF_n = F_{-1} + \sum_{i=2}^{t} F_{n+c_i} = F_0 + \sum_{i=2}^{t} F_{n+c_i}.$$

Applying Lemma 3, we derive $n + c_2 \geq 2$. Thus, the right-hand side is the Zeckendorf representation of kF_n and we obtain $f(kF_n) = t$. Let $n = U(k) - 2$. Theorem 1 yields

$$kF_n = F_{-2} + \sum_{i=2}^{t} F_{n+c_i} = \sum_{i=2}^{t} F_{n+c_i}. \tag{2.14}$$

The right-hand side of (2.14) is the Zeckendorf representation of kF_n; hence, $f(kF_n) = t-1$ and the proof is complete. □

Remark: To see that $R(k)$ is the same as in Filipponi's Conjecture 2, note that $\mu = 2j-1$ if $L_{2j-1} < k \leq L_{2j+1}$ and if $F_\nu + F_{\nu-6}$ (in the definition of ν) can be replaced by $2L_{\nu-3}$.

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