FIBONACCI EXPANSIONS AND "F-ADIC" INTEGERS

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I. INTRODUCTION

A Fibonacci expansion of a nonnegative integer n is an expression of n as a sum of Fibonacci numbers F_k with $k \geq 2$. It may be thought of as a partition of n into Fibonacci parts. The most commonly studied such expansion is the unique one in which the parts are all distinct and no two consecutive Fibonacci numbers appear. C. G. Lekkerkerker first showed this expansion was unique in 1952 [5]. There is also a unique dual form of this expansion in which no two consecutive Fibonacci numbers not exceeding n do *not* occur in the expansion [2]. Lekkerkerker's expansion is the only one I refer to in the remainder of this paper; from now on, I will call it the Fibonacci expansion of n, or fib(n) (I will give a precise definition in Part II). The Fibonacci expansion of nonnegative integers is similar in many ways to a fixed-base expansion (in fact, in some sense, it may be thought of as a base-τ expansion, where $\tau = \frac{1}{2}(1+\sqrt{5}) \approx 1.61803$ is the golden mean). First, in each case there are both "top-down" and "bottom-up" algorithms for obtaining the expansion of a nonnegative integer (see [3], pages 281-282). Second, there are mechanical rules for adding the expansions of two or more nonnegative integers [1]. Third, each case may be generalized by defining infinite expansions (p-adic or "F-adic" integers), both of which have interesting algebraic properties. One should be warned, however, that this analogy has its limitations. For instance, the p-adic integers form a ring, but the F-adic integers do not. My main result in this paper is that there is a 1-1 correspondence between the F-adic integers and the points on a circle, and that both of these sets share some important geometric properties.

II. FIBONACCI EXPANSIONS OF NONNEGATIVE INTEGERS

Definition: Let $n \in \omega = \{0, 1, 2, \ldots\}$. Suppose there exists a sequence $(c_k)_0^\infty \in \{0, 1\}^\omega$ such that $c_k c_{k+1} = 0\ (\forall k)$ and $n = \sum_{k=0}^\infty c_k F_{k+2}$. Then (c_k) is called the *Fibonacci expansion of* n and is denoted fib(n). It is well known that every nonnegative integer has a unique Fibonacci expansion [5], so fib: $\omega \to \{0, 1\}^\omega$ is well defined.

In this paper, I use the convention of increasing coefficient indices in Fibonacci expansions going from left to right. Thus, for instance,

$$\begin{aligned} \text{fib}(5) &= 0001 \\ \text{fib}(10) &= 01001 \\ \text{fib}(100) &= 0010100001 \end{aligned}$$

where the rightmost 1 in each expansion is assumed to be followed by an infinite sequence of zeros.

The top-down algorithm for computing fib(n) is as follows (see [4], page 573). First, find the largest nonnegative integer k such that F_{k+2} does not exceed n, and let $c_k = 1$. Next, subtract F_{k+2} from n and repeat the above procedure for the difference. After a finite number of iterations, the difference will be zero; then we have obtained fib(n). This procedure is well known, and it is easy to check that the resulting expansion has the right form (see [3], [5]).

As an example, suppose $n = 10$. Since $F_6 = 8$ is the largest Fibonacci number not exceeding 10, we set $c_4 = 1$ and subtract 8 from 10, obtaining 2. Since $2 = F_3$, we set $c_1 = 1$; now our difference is $2 - 2 = 0$, so we stop. Thus, fib(10) = 01001.

III. BOTTOM-UP ALGORITHM

Both top-down and bottom-up algorithms for expanding a nonnegative integer in a fixed base are well known. For instance, to find the binary expansion of a nonnegative integer n, we could proceed by first finding the largest power k of 2 less than n, setting $c_k = 1$, subtracting 2^k from n, and repeating this procedure until $n = 0$. This is the top-down algorithm. Alternatively, we could first determine n mod 2, set this equal to c_i for $i = 0$, subtract c_i from n, divide n by 2, increase i by 1, and repeat until $n = 0$. This is the bottom-up algorithm. The top-down algorithm given for finding fib(n) is clearly analogous to the top-down procedure for finding the binary expansion of n. By analogy with the binary case, we look for a bottom-up algorithm for calculating fib(n). Such an algorithm does exist; moreover, this algorithm makes it clear how to extend the Fibonacci expansion to negative integers and, more generally, "F-adic" integers. The algorithm goes as follows:

Step 1: Let $i = 0$.
Step 2: Let x be the unique real number congruent to n mod τ^2 and lying in the interval $[-1, \tau)$. Determine whether x lies in the closed interval from $-(-\tau)^{-i-1}$ to $-(-\tau)^{-i}$. (Note that the intervals zero in on the origin as i increases.) If x lies in the subinterval, let $c_i = 0$ and increase i by 1; otherwise, let $c_i = 1$, $c_{i+1} = 0$, decrease n by F_{i+2}, and increase i by 2.
Step 3: If $n = 0$, stop. Otherwise, go to Step 2.

Again, I illustrate for $n = 10$. It is straightforward to check that the unique real number in the interval $[-1, \tau)$ congruent to 10 mod τ^2 is $x = 10 - 4\tau^2 \approx -0.47213$. Since $x \in [-1, \tau^{-1}]$, we have $c_0 = 0$. Thus, we leave n alone and increase i to 1. Now we check whether x lies in $[-\tau^{-2}, \tau^{-1}]$; it does not, since the lower limit is too high. Thus, we set $c_1 = 1$, decrease n by $F_3 = 2$ to 8, and increase i by 2 to 3. Now $8 - 3\tau^2 \approx 0.14590$, which lies in the interval $[-\tau^{-4}, \tau^{-3}]$. (As we will see shortly, by Lemma 1, we have $8 = F_6 \equiv \tau^{-4} \approx 0.14590$.) Thus, we set $x = \tau^{-4}$, $c_2 = c_3 = 0$, and increase i by 1, leaving $n = 8$ alone. Next, we check whether x lies in the interval $[-\tau^{-4}, \tau^{-5}]$. It does not, so we set $c_4 = 1$ and decrease n by $F_6 = 8$. But now $n = 0$, so we stop. Thus, we have again obtained fib(10) = 01001.

To show that the above algorithm works, we need a few lemmas.

Lemma 1: $F_k \equiv (-\tau)^{2-k} \bmod \tau^2 \ (\forall k \in \omega)$.

Proof: Since $F_0 = 0 \equiv \tau^2 \bmod \tau^2$ and $F_1 = 1 \equiv -\tau \bmod \tau^2$, the lemma holds for $k = 0$ and 1. Also, note that $(-\tau)^{2-k} + (-\tau)^{1-k} = (-\tau)^{-k}(\tau^2 - \tau) = (-\tau)^{-k}$. The lemma then follows by induction on k. □

Lemma 2: Let $\overline{n} = \sum_{k=0}^{\infty} c_k(-\tau)^{-k}$, where the c_k's are the coefficients of the Fibonacci expansion of n. Then $\overline{n}$ is the unique real number in the interval $[-1, \tau)$ congruent to n mod τ^2.

Proof: Let $n=\sum_{k=0}^{\infty}c_kF_{k+2}\equiv\sum_{k=0}^{\infty}c_k(-\tau)^{-k}=\overline{n}\ (\text{mod}\ \tau^2)$. Thus, it is enough to show that $-1\le\overline{n}<\tau$ (uniqueness then follows, since the interval $[-1,\tau)$ has length τ^2). The supremum of $\overline{n}$ is attained by setting $c_k=1$ for all even k and 0 for odd k; its value is $1+\tau^{-2}+\tau^{-4}+\cdots=\tau$. Similarly, the infimum of $\overline{n}$ is $(-\tau^{-1})+(-\tau^{-3})+(-\tau^{-5})+\cdots=-1$. □

Lemma 3: $c_k=0\ (\forall k<\ell)\Leftrightarrow\overline{n}\in\begin{cases}(-\tau^{-\ell},\tau^{1-\ell}) & \ell\text{ even},\\(-\tau^{1-\ell},\tau^{-\ell}) & \ell\text{ odd.}\end{cases}$

Proof: First, I will prove the forward implication. Consider the case where ℓ is even (the case where ℓ is odd is similar). Clearly, $\overline{n}<\tau^{-\ell}+\tau^{-\ell-2}+\tau^{-\ell-4}+\cdots=\tau^{1-\ell}$. (The upper limit is approached by an arbitrarily long string of alternating 1's and 0's in fib(n) with the leading 1 in the ℓ^{th} position.) Similarly, $\overline{n}>-\tau^{-\ell-1}-\tau^{-\ell-3}-\cdots=-\tau^{-\ell}$.

The proof of the reverse implication goes as follows. Let k be the smallest integer such that $c_k=1$ and assume $k<\ell$. Let $n'=n-F_{k+2}$. Clearly, the first $k+1$ coefficients of fib(n') are zero, so by the first part of the proof (replacing ℓ by $k+2$), we get

$$\overline{n'}\in\begin{cases}(-\tau^{-(k+2)},\tau^{-(k+1)}) & k\text{ even},\\(-\tau^{-(k+1)},\tau^{-(k+2)}) & k\text{ odd.}\end{cases}$$

First, suppose k is even. Then $\overline{n}=\overline{n'}+\tau^{-k}$ lies in $(\tau^{-(k+1)},\tau^{-(k-1)})$. But then $\overline{n}$ is too big to fall in any interval of the form $(-\tau^{-\ell},\tau^{-\ell})$ for $\ell>k$, so the inverse implication holds. A similar argument can be used in the case where k is odd. □

First, note that $n\equiv\overline{n}\bmod\tau^2$ by Lemma 1; thus, if there exists an integer m such that $n-m\tau^2$ lies in the closed interval from $(-\tau)^{-i-1}$ to $(-\tau)^{-i}$, then; $c_k=0$ for $k\le i$ by Lemma 2. This justifies the first if-then statement of Step 2 of the algorithm. If the condition in Step 2 is not met, c_i must be 1; in this case, we subtract F_{i+2} from n, obtaining a new n with $c_k=0$ for $k<i+2$; this justifies the second if-then statement of Step 2. Thus, the algorithm works.

The basis of the bottom-up algorithm is the fact that, if m mod τ^2 and n mod τ^2 are close (here, m and n are nonnegative integers), then the first few coefficients of fib(m) and fib(n) are the same. Figure 1 illustrates this. On the left side of the figure, n and fib(n) are plotted and tabulated against $\overline{n}$ (height along the figure) for $0\le n\le 21=F_8$. Although the figure is illustrated as a vertical line, it should be thought of as a circle with circumference τ^2. See Part V for an explanation of the right side of the figure.

IV. ADDITION OF FIBONACCI EXPANSIONS OF NONNEGATIVE INTEGERS

Here, I present an algorithm for adding two Fibonacci expansions of nonnegative integers (see [4], [5]); i.e., given $\text{fib}(m)=(a_k)$ and $\text{fib}(n)=(b_k)$, it finds $\text{fib}(m+n)=(c_k)$. The algorithm goes as follows. First, add the expansions coefficientwise, i.e., let $c_k=a_k+b_k$ for all k. The result will be a string of 0's, 1's, and 2's. To get rid of the 2's and consecutive 1's, apply the transformations

$$x+1,\ y+1,\ 0\mapsto x,\ y,\ 1$$
$$x,\ 0,\ y+2,\ 0\mapsto x+1,\ 0,\ y,\ 1$$

fib(n)	n
101010	12
101000	4
101001	17
100010	9
100000	1
100001	14
100100	6
100101	19
001010	11
001000	3
001001	16
000010	8
000000	0
000001	13
000100	5
000101	18
010010	10
010000	2
010001	15
010100	7
010101	20

$\overline{n} = \tau$

n	$\text{fib}_1(n)$	$\text{fib}_2(n)$
-1	$0101010\overline{10}$	$1010101\overline{01}$
-9	$1010001\overline{01}$	$1010100\overline{10}$
-17	$1010000\overline{10}$	$1010010\overline{01}$
-4	$1000101\overline{01}$	$1010010\overline{10}$
-12	$1000001\overline{01}$	$1000100\overline{10}$
-20	$1000000\overline{10}$	$1000010\overline{01}$
-7	$1000010\overline{10}$	$1001001\overline{01}$
-15	$1001000\overline{10}$	$1001010\overline{01}$
-2	$0010101\overline{01}$	$1001010\overline{10}$
-10	$0010001\overline{01}$	$0010100\overline{10}$
-18	$0010000\overline{10}$	$0010010\overline{01}$
-5	$0000101\overline{01}$	$0010010\overline{10}$
-13	$0000001\overline{01}$	$0000100\overline{10}$
-21	$0000000\overline{10}$	$0000010\overline{01}$
-8	$0000010\overline{10}$	$0001001\overline{01}$
-16	$0001000\overline{10}$	$0001010\overline{01}$
-3	$0001010\overline{10}$	$0100101\overline{01}$
-11	$0100001\overline{01}$	$0100100\overline{10}$
-19	$0100000\overline{10}$	$0100010\overline{01}$
-6	$0100010\overline{10}$	$0101001\overline{01}$
-14	$0101000\overline{10}$	$0101010\overline{01}$
-1	$0101010\overline{10}$	$1010101\overline{01}$

$\overline{n} = -1$

FIGURE 1

to the rightmost applicable string. Continue until (c_k) has no 2's or consecutive 1's. These transformations are justified by the identities $F_{k+2} = F_{k+1} + F_k$ and $2F_k = F_{k+1} + F_{k-2}$. The algorithm must terminate after a finite number of steps, since each step increases the value of (c_k) viewed as a ternary number with the order of the digits reversed, and this cannot increase indefinitely because the last digit must correspond to a Fibonacci number that does not exceed $m+n$. However, it should be noted that, as presented, this algorithm is not complete, since it may yield an expansion (c_k) with a nonzero coefficient for $k=-1$ or $k=-2$. For instance, adding 1 and 1 gives the expansion 10.01 for 2 (coefficients with negative indices appear to the left of the decimal point). The case $k=-2$ is easy to deal with; simply eliminate this coefficient. This can be done because $F_0 = 0$. In the case where $c_{-1} = 1$, first set $c_{-1} = 0$ and $c_0 = 1$. (In this case, c_0 must have been 0 previously, since we have no two consecutive 1's at this stage.) Next, apply the first transformation repeatedly, this time starting on the left, until (c_k) is in the standard form. Again, only a finite number of applications is necessary, since each one decreases the number of 1's by 1.

V. NEGATIVE AND "F-ADIC" INTEGERS

One advantage of the bottom-up algorithm is that it allows a straightforward extension of fib to negative integers. We run the algorithm as stated, but must now allow for infinite expansions. For instance, applying the algorithm to -1, we get fib$(-1) = 01010101\ldots$. Note that, if we had used open instead of closed intervals in Step 2, we would have obtained fib$(-1) = 10101010\ldots$.

In fact, both expansions are valid, and these are the only ones. This dichotomy occurs for all negative integers. We use the notation fib_1 for the first case (open intervals) and fib_2 for the other case (closed intervals). Note that using closed intervals gives priority to the first 0 in the Fibonacci expansion where there is a choice between a 0 or a 1. Thus, the first coefficient that differs in $\text{fib}_1(n)$ and $\text{fib}_2(n)$ is a 0 in the former and a 1 in the latter. This can be seen in Figure 1 above.

Although the bottom-up algorithm as stated can be used to find $\text{fib}_1(n)$ for all integers n, it is not practical to use it directly for negative integers. A better method is as follows. First, find the smallest Fibonacci number $F_{k+2} \geq -n$. Set the first $k+1$ coefficients of $\text{fib}_1(n)$ equal to those of $\text{fib}(F_{k+2}+n)$. Finally, for $i > k$, set $c_i = 0$ if i and k have the same parity; otherwise, set $c_i = 1$. For example, say $n = -24$. Then the smallest Fibonacci number exceeding $-n$ is $F_9 = 34$, so we set the first eight coefficients of fib(-24) equal to those of $\text{fib}(34-24) = \text{fib}(10)$, i.e., 01001000. For $i \geq 8$, we set $c_i = 0$ if i is odd; 1 otherwise. Thus, $\text{fib}_1(-24) = 01001000\overline{10}$, where, as in the case for repeating decimal expansions, a line above a string of coefficients means that string is repeated endlessly.

What about $\text{fib}_2(n)$? It can be found by a simple modification of the above procedure. First, instead of finding the smallest Fibonacci number exceeding $-n$, find the next smallest; in the above example, this would be $F_{10} = 55$. Again, calculate $\text{fib}(F_{k+2}+n)$, and set the first $k+1$ coefficients of fib(n) equal to these. (Now, however, k is one greater than last time.) Thus, returning to the example, $\text{fib}(55-24) = \text{fib}(31) = 010010100$. The last step is exactly the same as before, but with k replaced by $k+1$; thus, $\text{fib}_2(-24) = 01001010\overline{01}$ is the other expansion. Note that one expansion has $c_k = 0$ for even k in the repeating portion of fib(n), and the other has $c_k = 0$ for odd k. This is always the case. Also, note that the nonrepeating portions of the two expansions only differ in one place. In fact, for all negative integers except -1, the nonrepeating portions differ in one place. (The two expansions of -1 are both purely periodic.)

Let us refer again to the right side of Figure 1. Note that the negative integers lie on the borderline of regions where c_i is constant for $i \leq k$ for some k. Also note that the positions of the positive and negative integers are staggered and that, for negative n, the first six coefficients of $\text{fib}_1(n)$ and $\text{fib}_2(n)$ agree with the expansions of the two positive neighbors of n.

The bottom-up algorithm involves first calculating the residue of an integer mod τ^2. What if, instead, we start with an arbitrary real number x, calculate its residue class mod τ^2 and apply the bottom-up algorithm? Then we will, in general, obtain an infinite sequence of 0's and 1's with no two consecutive 1's. Let U be the set of all such sequences, and define the equivalence relation E by letting two distinct sequences in U be equivalent iff one is $\text{fib}_1(n)$ and the other is $\text{fib}_2(n)$ for some negative integer n. Define the *F-adic integers* to be the elements of the set U/E (they are analogous to p-adic integers).

VI. GEOMETRIC STRUCTURE OF "F-ADIC" INTEGERS

In Figure 1 I illustrated how the Fibonacci expansions of the integers have a nice interpretation as points on a circle of circumference τ^2. In this section I would like to make that analogy more precise and to extend it to all the F-adic integers, which I denote $\mathbf{Z}_F$. (In fact, one can show that $\mathbf{Z}_F$ is a topological group isomorphic to the circle group, but the proof is rather unenlightening.)

As indicated in Part V, the bottom-up algorithm may be applied to any arbitrary real number x to give an F-adic integer, which is unique up to the congruence class of x mod τ^2. Thus, there is a 1-1 correspondence between the F-adic integers and points on a circle of circumference τ^2. Furthermore, as can be seen in Figure 1, nearby points on the circle seem to correspond to "nearby" F-adic integers, where "nearby" roughly means having Fibonacci expansions agreeing to the first several places. I first need to make the notion of "nearby" F-adic integers more precise.

Definition: Let α and β be F-adic integers. Then α and β are *similar in k places* if there exists an F-adic integer γ and sequences $(a_i), (b_i), (c_i)$, and (c_i') in U such that $\alpha \sim (a_i)$, $\beta \sim (b_i)$, $(c_i) \sim (c_i')$, and for all $i < k$, we have $a_i = c_i$ and $b_i = c_i'$. Here, $\sim$ denotes the equivalence defined in Part V. If α and β are similar in m places but not in $m+1$ places, we say they are similar in *exactly m* places.

For example, suppose $\alpha = 4$ and $\beta = 20$. Then $(a_i) = \text{fib}(4) = 101\overline{0}$ and $(b_i) = \text{fib}(20) = 010101\overline{0}$. Let $\gamma = -1$ and let $(c_i) = \text{fib}_2(-1) = \overline{10}$ and $(c_i') = \text{fib}_1(-1) = \overline{01}$. Then $a_i = c_i$ and $b_i = c_i'$ for $i < 4$ so α and β are similar in four places.

Now I will state and prove the main theorem of my paper.

Theorem: There exists a bijection $\phi: \mathbf{Z}_F \to \mathbf{R} / \tau^2\mathbf{Z}$ for which, given any pair of F-adic integers α and β which are similar in k places, there exists a real number $x \equiv \phi(\alpha) - \phi(\beta) \pmod{\tau^2}$ such that $|x| \le 2\tau^{2-k}$. Conversely, if $\alpha, \beta \in \mathbf{Z}_F$ are such that there exists a real number $x \equiv \phi(\alpha) - \phi(\beta)$ (mod τ^2) such that $|x| \le \tau^{-k}$, then α and β are similar in k places.

Proof: Consider the map

$$\hat{\phi}: U \to \mathbf{R} / \tau^2\mathbf{Z}$$

$$(c_i)_0^\infty \mapsto \sum_{i=0}^{\infty} c_i(-\tau)^{-i} \pmod{\tau^2}.$$

Note that the inverse of $\hat{\phi}$ is just the bottom-up algorithm, and that this inverse is unique except when (c_i) corresponds to a negative integer. Thus, we may define $\phi(x)$ to be $\hat{\phi}(\bar{x})$, where $\bar{x}$ is the equivalence class of x in $\mathbf{Z}_F$.

Now let α and β be F-adic integers that are similar in k places. Then there exist sequences $\alpha \sim (a_i)$ and $\beta \sim (b_i)$ and an F-adic integer $\gamma \sim a_0 a_1 \ldots a_{k-1} c_k c_{k+1} \ldots \sim b_0 b_1 \ldots b_{k-1} c_k' c_{k+1}' \ldots$. Now let $\alpha' = a_0 \ldots a_{k-1}\overline{0}$ and $\beta' = b_0 \ldots b_{k-1}\overline{0}$. By Lemma 3, both $\phi(\alpha) = \phi(\alpha')$ and $\phi(\gamma) - \phi(\alpha')$ lie in a fixed interval of the form $\pm[-\tau^{-k}, \tau^{1-k}]$, so their difference, $\phi(\alpha) - \phi(\gamma)$, has absolute value not exceeding the length of the interval, τ^{2-k}. Similarly, $|\phi(\beta) - \phi(\gamma)| \le \tau^{2-k}$. Thus, by the triangle inequality, $|\phi(\alpha) - \phi(\beta)| \le 2\tau^{2-k}$.

To prove the converse, suppose α and β are as in the statement of the second half of the theorem. Say $\alpha \sim (a_i)$ and $\beta \sim (b_i)$. Suppose (a_i) and (b_i) agree to exactly ℓ places, so that $a_i = b_i = c_i$ for $i < \ell$, and $a_\ell \ne b_\ell$. Without loss of generality, we may assume $a_\ell = 0$ and $b_\ell = 1$. Note that $c_{\ell-1} = 0$ since, otherwise, (b_i) would contain two consecutive 1's. Let n be the unique negative integer such that $\text{fib}_1(n)$ agrees with (a_i) to $\ell+1$ places and $\text{fib}_2(n)$ agrees with (b_i) also to $\ell+1$ places. Now we have

$$\text{fib}_1(n) = c_0c_1c_2\dots c_{\ell-1}0\overline{01}$$
$$\text{and } \text{fib}_2(n) = c_0c_1c_2\dots c_{\ell-1}10\overline{01}.$$

Now suppose α and β are similar *through* n in exactly m places, where $m > \ell$, i.e., γ may be replaced by n in the definition of similarity. Then there are two possibilities for what (a_i) and (b_i) look like, depending on whether (a_i) or (b_i) has the first discrepancy from $\text{fib}_1(n)$ or $\text{fib}_2(n)$, respectively. In the first case, we have

$$(a_i) = c_0c_1c_2\dots c_{\ell-1}001\dots0100a_{m+1}a_{m+2}\dots,$$
$$(b_i) = c_0c_1c_2\dots c_{\ell-1}1001\dots010b_{m+1}b_{m+2}\dots,$$

where the second string of dots in each expansion stands for a finite repeating string of the form $01\dots01$. Note that in this case, (b_i) necessarily agrees with $\text{fib}_2(n)$ to at least $m+1$ places and that $m \equiv \ell \pmod 2$. From the definition of ϕ, we have

$$\begin{aligned}(-1)^m(\phi(n)-\phi(\alpha)) &\equiv \tau^{-m} + a_{m+1}\tau^{-m-1} \\ &\quad + (1-a_{m+2})\tau^{-m-2} + a_{m+3}\tau^{-m-3} + \cdots \\ &\geq \tau^{-m}\end{aligned}$$

and

$$\begin{aligned}(-1)^m(\phi(\beta)-\phi(n)) &\equiv (1-b_{m+1})\tau^{-m-1} + b_{m+2}\tau^{-m-2} \\ &\quad + (1-b_{m+3})\tau^{-m-3} + b_{m+4}\tau^{-m-4} + \cdots \\ &\geq 0,\end{aligned}$$

where the congruence is modulo τ^2.

In the second case, we have

$$(a_i) = c_0c_1c_2\dots c_{\ell-1}001\dots01010a_{m+1}a_{m+2}\dots,$$
$$(b_i) = c_0c_1c_2\dots c_{\ell-1}1001\dots0100b_{m+1}b_{m+2}\dots.$$

This time, we see that (a_i) necessarily agrees with $\text{fib}_1(n)$ to at least $m+1$ places and that $\ell \not\equiv m \pmod 2$. Now we find

$$\begin{aligned}(-1)^m(\phi(n)-\phi(\alpha)) &\equiv (1-a_{m+1})\tau^{-m-1} + a_{m+2}\tau^{-m-2} \\ &\quad + (1-a_{m+3})\tau^{-m-3} + a_{m+4}\tau^{-m-4} + \cdots \\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}(-1)^m(\phi(\beta)-\phi(n)) &\equiv \tau^{-m} = b_{m+1}\tau^{-m-1} \\ &\quad + (1-b_{m+2})\tau^{-m-2} + b_{m+3}\tau^{-m-3} + \dots \\ &\geq \tau^{-m}.\end{aligned}$$

Again, the congruence is modulo τ^2. In each case, since $\phi(\gamma)$ is between $\phi(\alpha)$ and $\phi(\beta)$, we conclude

$$|\phi(\alpha)-\phi(\beta)| = |\phi(\beta)-\phi(\gamma)| + |\phi(\gamma)-\phi(\alpha)| \geq \tau^{-m},$$

where the above absolute values refer to the minimal such absolute values of real numbers belonging to the congruence class of the expression inside modulo τ^2. But since we are assuming $|x| \leq \tau^{-k}$ for some real number $x \equiv (\phi(\alpha)-\phi(\beta)) \pmod \tau$, we conclude that $m \geq k$. Thus, α and β are similar in k places. □

As I indicated earlier, the F-adic integers have more structure than I have presented. For instance, the map ϕ may be used to define addition on $\mathbf{Z}_F$. This addition makes $\mathbf{Z}_F$ into an additive group isomorphic to the circle group [i.e., by requiring that $\phi(\alpha+\beta)=\phi(\alpha)+\phi(\beta)$.] The map ϕ also turns out to be a topological group isomorphism.

VII. GENERALIZATIONS

There are many ways to generalize the above procedure to other types of sequences. Perhaps the simplest (see [4]) is to consider sequences of the form $S_{k+1}=aS_k+bS_{k-1}$; $S_1=S_2=1$, where a and b are positive integers with $a\ge b$. The corresponding expansion is $E(n)=(e_k)_{k=0}^{\infty}$, where $n=\sum_{k=0}^{\infty}e_kS_{k+2}$, where now $0\le e_k\le a$ and $e_k=a$ implies $e_{k+1}<b$. Let $\lambda=\frac{1}{2}(a+\sqrt{a^2+4b})$ and $\overline{\lambda}=\frac{1}{2}(a-\sqrt{a^2+4b})$; then it is easy to check that $S_{k+2}\equiv\overline{\lambda}^k \bmod \lambda$. Since $|\overline{\lambda}|<1$, $S_k\to 0 \bmod \lambda$; thus, we should again have a bottom-up algorithm for determining $E(n)$. It looks like the same analysis should carry through for these more general sequences. In particular, if we again define the analogous infinite sequences of coefficients (e_k), they should again form an additive group isomorphic to $\mathbf{R}/\mathbf{Z}$. One can also carry out this procedure for a much more general class of sequences. The reader is invited to try his hand with the sequence 1, 10, 100, 1000, ...

Another way to generalize is to define "F-adic numbers," the analog of p-adic numbers. At first, this does not seem feasible, for the F_n are integers for negative as well as positive n, so we gain nothing by considering sums of the form $\sum_{k=\ell}^{\infty}c_kF_{k+2}$, where $\ell<0$. The solution is to just consider formal sequences of the form $(c_k)_{\ell}^{\infty}$, where $c_k=0$ or 1, $c_kc_{k+1}=0$ for all k, and $\ell\in\mathbf{Z}$. We treat these sequences as before, but simplify the addition algorithm so as not to worry about fixing coefficients with negative indices. The resulting group seems to be isomorphic to $\mathbf{R}$. It should be noted (see [5]) that an ordinary integer n will, in general, have a *different* expansion of this type than fib(n).

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