ON THE STRUCTURE OF QUADRATIC IRRATIONALS ASSOCIATED WITH GENERALIZED FIBONACCI AND LUCAS NUMBERS

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(Submitted July 1994)

1. INTRODUCTION

In 1970 C. T. Long and J. H. Jordan completed a series of two papers, [3] and [4], in which they analyzed the arithmetical structure of certain classes of quadratic irrationals and the effects on their structure after multiplication by rational numbers. In particular, for a positive integer a, let $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ be the n^{th} *generalized Fibonacci* and *generalized Lucas numbers*, respectively. That is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, $\mathcal{L}_0 = 2$, $\mathcal{L}_1 = a$ and, for $n > 1$, $\mathcal{F}_n = a\mathcal{F}_{n-1} + \mathcal{F}_{n-2}$, $\mathcal{L}_n = a\mathcal{L}_{n-1} + \mathcal{L}_{n-2}$. We denote the *generalized golden ratio* by φ_a. Thus,

$$\varphi_a = \frac{a + \sqrt{a^2 + 4}}{2} = [a, a, \dots] = [\overline{a}],$$

where the last expression denotes the (simple) continued fraction expansion for φ_a and the bar indicates the periodicity. It follows that $\lim_{n\to\infty} \mathcal{F}_{n+1}/\mathcal{F}_n = \varphi_a$. We note that in the case in which $a = 1$ we have $\mathcal{F}_n = F_n$, $\mathcal{L}_n = L_n$, and $\varphi_a = \varphi$.

Among their other interesting results, Long and Jordan investigated and compared the continued fraction expansions of $\frac{r}{s}\varphi_a$ and $\frac{s}{r}\varphi_a$ when r and s are consecutive generalized Fibonacci numbers or consecutive generalized Lucas numbers. These results led them to consider the structure of numbers of the form $\frac{r}{s}\varphi_a$ and $\frac{s}{r}\varphi_a$ where $r = \mathcal{F}_n$ and $s = \mathcal{L}_n$. They wrote (in the present notation) [4]:

"In view of the preceding results, one would expect an interesting theorem concerning the simple continued fraction of

$$\frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a \quad \text{and} \quad \frac{\mathcal{L}_n}{\mathcal{F}_n}\varphi_a$$

but we were unable to make a general assertion value for all a. To illustrate the difficulty, note that, when $a = 2$ and $\varphi_2 = 1 + \sqrt{2}$, we have

$$\frac{\mathcal{F}_4}{\mathcal{L}_4}\varphi_2 = \left[0, 1, \overline{5, 1, 3, 5, 1, 7}\right],$$

$$\frac{\mathcal{F}_5}{\mathcal{L}_5}\varphi_2 = \left[0, 1, \overline{5, 1, 5, 3, 1, 4, 1, 7}\right],$$

$$\frac{\mathcal{F}_6}{\mathcal{L}_6}\varphi_2 = \left[0, 1, \overline{5, 1, 4, 1, 3, 5, 1, 4, 1, 7}\right]."$$

They do, however, discover the following two beautiful identities for the case in which $a = 1$. We state them here as

Theorem 1: For $n \geq 4$,

$$\frac{F_n}{L_n}\varphi = \left[0, 1, \overline{2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 3, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 4}\right] \tag{1.1}$$

and

$$\frac{L_n}{F_n}\varphi = \left[3, \overline{\underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 3, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 2, 4}\right].$$

Our first objective here is to extend their result to the more general case for arbitrary a. We begin with the elementary observation that

$$\lim_{n\to\infty} \frac{F_n}{L_n}\varphi = \frac{5+\sqrt{5}}{10} = \left[0, 1, 2, \overline{1}\right],$$

and, thus, the initial string of 1's in (1.1) is not surprising. It is the beautiful near mirror symmetry of the interior portion of the periodic part of (1.1) that is unexpected. More generally, one has

$$\lim_{n\to\infty} \frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a = \frac{a^2+4+a\sqrt{a^2+4}}{2(a^2+4)} = \left[0, 1, a^2+1, \overline{1, a^2}\right].$$

Thus, for large n, we would expect the continued fraction expansion for $(\mathcal{F}_n / \mathcal{L}_n)\varphi_a$ to begin with $[0, 1, a^2+1, 1, a^2, 1, a^2, 1, \ldots]$. As Long and Jordan remark, however, in this case we appear to lose the symmetry. In fact, the near mirror symmetry in (1.1) is somewhat deceptive. Perhaps it is better to view (1.1) as a "recursive system" in the following sense. We define the strings or "words" $\vec{W}_n = W_n$ for $n \geq 4$ by $W_4 = (3, 1)$ and, for $n > 4$, $W_n = (\overleftarrow{W}_{n-1}, 1, 1)$, where $\overleftarrow{W}_{n-1}$ is the word W_{n-1} read *backwards*. For example, $W_5 = (1, 3, 1, 1)$ and $W_6 = (1, 1, 3, 1, 1, 1)$. Thus, we may now reformulate (1.1) as

$$\frac{F_n}{L_n}\varphi = \left[0, 1, \overline{2, \vec{W}_n, 4}\right].$$

We note that the continued fraction expansions given above for $(\mathcal{F}_n / \mathcal{L}_n)\varphi_a$ obey a similar recursive behavior. This leads to our first result.

Theorem 2: Let $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ be the n^{th} generalized Fibonacci and Lucas numbers, respectively. Let $\vec{\mathcal{W}}_4 = \mathcal{W}_4(a) = (1, a^2-1, a^2+1, 1)$ and, for $n > 4$, let $\vec{\mathcal{W}}_n = \mathcal{W}_n(a) = (\overleftarrow{\mathcal{W}}_{n-1}, a^2, 1)$. Then, for $n \geq 4$,

$$\frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a = \left[0, 1, \overline{a^2+1, \vec{\mathcal{W}}_n, a^2+3}\right], \tag{1.2}$$

and

$$\frac{\mathcal{L}_n}{\mathcal{F}_n}\varphi_a = \left[a^2+2, \overline{\overleftarrow{\mathcal{W}}_n, a^2+1, a^2+3}\right]. \tag{1.3}$$

We remark that for $a = 1$,

$$[\mathcal{W}_4] = 1 + \cfrac{1}{0 + \cfrac{1}{2 + \cfrac{1}{1}}} = 1 + 2 + \frac{1}{1} = 3 + \frac{1}{1} = [W_4],$$

thus, Theorem 1 and Theorem 2 are equivalent when $a = 1$.

One may define the words $\vec{\mathcal{W}}_n$ occurring in Theorem 2 explicitly rather than iteratively. In particular, a simple induction argument reveals that, for $n \geq 4$ even,

$$\vec{\mathcal{W}}_n = \left(\{1, a^2\}^{(n-4)/2}, 1, a^2 - 1, a^2 + 1, 1, \{a^2, 1\}^{(n-4)/2}\right), \tag{1.4}$$

and, for $n > 4$ odd,

$$\vec{\mathcal{W}}_n = \left(\{1, a^2\}^{(n-5)/2}, 1, a^2 + 1, a^2 - 1, 1, \{a^2, 1\}^{(n-3)/2}\right),$$

where by $\{1, a^2\}^n$ we mean the word $(1, a^2)$ repeated n times.

As Long and Jordan implicitly note with respect to Theorem 1, Theorem 2 immediately implies that $(\mathcal{F}_n / \mathcal{L}_n)\varphi_a$ and $(\mathcal{L}_n / \mathcal{F}_n)\varphi_a$ are not equivalent numbers. Recall that two real numbers are said to be *equivalent* if, from some point on, their continued fraction expansions agree (see [5]).

Next, we extend Theorem 1 in a different direction. We wish to analyze the structure of quadratic irrationals of the form $(\mathcal{F}_{mn} / \mathcal{L}_n)\varphi_a$. If m is even, then $\mathcal{F}_{mn} / \mathcal{L}_n$ is an integer (see [7]); thus, we consider only the case in which m is odd. We first state an extension of Theorem 1 in this context for the case $m = 3$.

Theorem 3: For $n \geq 4$, if n is even, then

$$\frac{F_{3n}}{L_n}\varphi = \left[F_{2n+1} - 1, \overline{3, \underbrace{1, 1, \ldots, 1, 1}_{n-2 \text{ times}}, L_{2n} - 2, 2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, 2, 1, L_{2n} - 2}\right].$$

If $n > 4$ is odd, then

$$\frac{F_{3n}}{L_n}\varphi = \left[F_{2n+1}, \overline{1, 2, \underbrace{1, 1, \ldots, 1, 1}_{n-4 \text{ times}}, L_{2n}, \underbrace{1, 1, \ldots, 1, 1}_{n-2 \text{ times}}, 3, L_{2n}}\right].$$

The general formulation of Theorem 3 appears to be more complicated and requires us to define several useful sums. For odd integers m, we let

$$\mathbf{F}_1(m) = \sum_{k=1}^{(m-1)/2} (-1)^k \mathcal{F}_{2k+1}, \qquad \mathbf{F}(m) = \sum_{k=1}^{(m-1)/2} \mathcal{F}_{2k+1},$$

$$\mathbf{L}_1(m) = \sum_{k=1}^{(m-1)/2} (-1)^k \mathcal{L}_{2k}, \qquad \mathbf{L}(m) = \sum_{k=1}^{(m-1)/2} \mathcal{L}_{2k}.$$

We remark that $\mathbf{F}_1(m)$ and $\mathbf{L}_1(m)$ are positive integers if and only if $m \equiv 1 \bmod 4$. We believe that one may generalize the proof of Theorem 3 to prove the following conjecture.

Conjecture 4: Let $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ be the n^{th} generalized Fibonacci and Lucas numbers, respectively, and m an odd integer. Suppose $m \geq 3$ is an odd integer and $n \geq 4$. For n even, let $\vec{\mathcal{U}}_n = \mathcal{U}_n(a) = (1, a^2+1, 1, \{a^2, 1\}^{(n-4)/2}, a^2-1, 1)$ and $\vec{\mathcal{V}}_n = \mathcal{V}_n(a) = (\{a^2, 1\}^{(n-2)/2}, a^2+2)$; for n odd, let $\vec{\mathcal{W}}_n = \mathcal{W}_n(a) = (\{a^2, 1\}^{(n-5)/2})$. If n is odd, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[\mathbf{F}(m), \overline{1, a^2+1, 1, \vec{\mathcal{W}}_n, a^2+1, a^2\mathbf{L}(m)+a^2-1, \vec{\mathcal{W}}_{n+2}, 1, a^2+2, \mathbf{L}(m)}\right].$$

If n is even and $m \equiv 1 \bmod 4$, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[\mathbf{F}_1(m), \overline{\vec{\mathcal{U}}_n, \mathbf{L}_1(m), \vec{\mathcal{V}}_n, \mathbf{L}_1(m)}\right].$$

If n is even and $m \equiv 3 \bmod 4$, then

$$\frac{\mathcal{F}_{mn}}{\mathcal{L}_n}\varphi_a = \left[-\mathbf{F}_1(m)-1, \overline{\vec{\mathcal{V}}_n, -\mathbf{L}_1(m)-2, \vec{\mathcal{U}}_n, -\mathbf{L}_1(m)-2}\right].$$

One may also find analogous expansions for $(\mathcal{L}_{mn}/\mathcal{F}_n)\varphi_a$. For example, one may adopt the method of proof of Theorem 3 to deduce

Theorem 5: For $n \ge 4$, if n is even, then

$$\frac{L_{3n}}{F_n}\varphi = \left[5F_{2n+1}+3, \overline{3, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 2, L_{2n}, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 2, 5L_{2n}+4}\right].$$

If $n > 4$ is odd, then

$$\frac{L_{3n}}{F_n}\varphi = \left[5F_{2n+1}-4, \overline{2, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, L_{2n}-2, 2, \underbrace{1, 1, \ldots, 1, 1}_{n-3 \text{ times}}, 5L_{2n}-6}\right].$$

Long and Jordan [4] concluded their investigation by proving the surprising result that, for any positive integers m and n, $(\mathcal{F}_m/\mathcal{F}_n)\varphi_a$ and $(\mathcal{F}_n/\mathcal{F}_m)\varphi_a$ are equivalent numbers. They remarked, however, that it is not always the case that $(\mathcal{L}_m/\mathcal{L}_n)\varphi_a$ and $(\mathcal{L}_n/\mathcal{L}_m)\varphi_a$ are equivalent numbers. To illustrate this, they noted that

$$\frac{L_2}{L_4}\varphi = [0, 1, \overline{2, 3, 1, 4}] \quad \text{and} \quad \frac{L_4}{L_2}\varphi = [3, \overline{1, 3, 2, 4}].$$

We observe that, in their example, the indices 2 and 4 are *not* relatively prime. Here we prove that this is the only possible case in which two such numbers are not equivalent. In particular, we prove

Theorem 6: If $\mathcal{L}_n = \mathcal{L}_n(a)$ is the n^{th} generalized Lucas number, then for relatively prime positive integers m and n,

$$\frac{\mathcal{L}_m}{\mathcal{L}_n}\varphi_a \quad \text{and} \quad \frac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a$$

are equivalent numbers.

More recently, Long [2] studied the arithmetical structure of classes of quadratic irrationals involving generalized Fibonacci and Lucas numbers of the form

$$\frac{aS_n + T_m\sqrt{a^2+4}}{2}, \tag{1.5}$$

where S_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ and T_m is either $\mathcal{F}_m$ or $\mathcal{L}_m$. For example, he investigated

$$\frac{a\mathcal{F}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}. \tag{1.6}$$

For numbers of the form (1.5), Long showed that their continued fraction expansions have the general shape

$$\left[a_0, \overline{a_1, a_2, \dots, a_r}\right],$$

where the a_0 and a_r were explicitly computed. He also proved that $(a_1, a_2, \dots, a_{r-1})$ is a palindrome, but was unable to determine the precise value of a_n for $0 < n < r$. Long also observed that the period length r appeared always to be even and that the value of r appeared not to be bounded as a function of a, m, and n. Here we claim that the continued fraction for such numbers may be completely determined. As an illustration, in Section 6 we provide the precise formula for the continued fraction expansion for numbers of the form (1.6). As the expansion is somewhat complicated in general, we do not state it here in the introduction; instead, we state it explicitly in Section 6 as Theorems 7 and 8. As a consequence of our results, we are able to prove that Long's first observation is true while his second observation is false.

2. BASIC IDENTITIES AND CONTINUED FRACTIONS

We begin with a list of well-known identities involving Fibonacci and Lucas numbers that will be utilized in our arguments (for proofs, see, e.g., [7]). For $n \geq 1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \tag{2.1}$$

$$F_n = F_{n-1} + F_{n-2},\ L_n = L_{n-1} + L_{n-2},\ L_n = F_{n+1} + F_{n-1}, \tag{2.2}$$

$$F_{n+m} - (-1)^m F_{n-m} = F_m L_n, \tag{2.3}$$

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n, \tag{2.4}$$

$$L_n^2 + 4(-1)^{n+1} = 5F_n^2, \tag{2.5}$$

$$F_{2n} = F_n L_n. \tag{2.6}$$

If $\mathcal{F}_n = \mathcal{F}_n(a)$ and $\mathcal{L}_n = \mathcal{L}_n(a)$ denote the n^{th} generalized Fibonacci and Lucas numbers, respectively, then for $n \geq 1$,

$$\begin{pmatrix} a^2+1 & a^2 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} \mathcal{F}_{2n+1} & a\mathcal{F}_{2n} \\ a^{-1}\mathcal{F}_{2n} & \mathcal{F}_{n-1} \end{pmatrix}, \tag{2.7}$$

$$\mathcal{L}_n = a\mathcal{F}_n + 2\mathcal{F}_{n-1}, \tag{2.8}$$

$$\mathcal{L}_n^2 = 4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2, \tag{2.9}$$

$$\mathcal{F}_n\mathcal{F}_{n+2} - \mathcal{F}_{n+1}^2 = (-1)^{n+1}, \tag{2.10}$$

$$\mathcal{F}_{n+2}\varphi_a + \mathcal{F}_{n+1} = \varphi_a(\mathcal{F}_{n+1}\varphi_a + \mathcal{F}_n), \tag{2.11}$$

$$(a^2+4)\mathcal{F}_n^2 + 4(-1)^n = \mathcal{L}_n^2. \tag{2.12}$$

For a real number α, we write $\alpha = [a_0, a_1, \ldots]$ for the simple continued fraction expansion of α. That is,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where all the a_n are integers and $a_n > 0$ for all $n > 0$ (for further details, see [5]). Basic to our method is a fundamental connection between 2×2 matrices and formal continued fractions. This connection has been popularized recently by Stark [6] and by van der Poorten [8] and [9]. Let $c_0, c_1, \ldots, c_N$ be real numbers. Then the *fundamental correspondence* may be stated as follows: If

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_N & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_N & p_{N-1} \\ q_N & q_{N-1} \end{pmatrix},$$

then

$$\frac{p_N}{q_N} = [c_0, c_1, \ldots, c_N].$$

We remark that since $c_0, c_1, \ldots, c_N$ are real numbers, p_N / q_N may not necessarily be rational.

3. THE PROOF OF THEOREM 2

We first consider the case in which n is even. Let α be the quadratic irrational defined by

$$\alpha = \left[\overline{a^2+1, \{1, a^2\}^{(n-4)/2}, 1, a^2-1, a^2+1, 1, \{a^2, 1\}^{(n-4)/2}, a^2+3}\right].$$

We will compute α via the fundamental correspondence between matrices and continued fractions. Thus, if we express the following matrix product as

$$\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}\right\}^{(n-4)/2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2-1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\cdot$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right\}^{(n-4)/2}\begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix},$$

then it follows that $\alpha = r/t$. In view of (2.7), we may express the above as

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{F}_{n-3} & a^{-1}\mathcal{F}_{n-4} \\ a\mathcal{F}_{n-4} & \mathcal{F}_{n-5} \end{pmatrix}\begin{pmatrix} a^2 & 1 \\ a^2-1 & 1 \end{pmatrix}\cdot$$

$$\begin{pmatrix} a^2+2 & a^2+1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{F}_{n-3} & a\mathcal{F}_{n-4} \\ a^{-1}\mathcal{F}_{n-4} & \mathcal{F}_{n-5} \end{pmatrix}\begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

The functional equation $\mathcal{F}_k = a\mathcal{F}_{k-1} + \mathcal{F}_{k-2}$ enables us to simplify the above product and carry out the multiplication to deduce

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 & \mathcal{F}_n^2 \\ (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 & \mathcal{F}_{n-1}^2 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2)\alpha + \mathcal{F}_n^2 & \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 \\ ((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha + \mathcal{F}_{n-1}^2 & (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 \end{pmatrix}$$

Thus, we have

$$\alpha = \frac{r}{t} = \frac{(\mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2)\alpha + \mathcal{F}_n^2}{((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha + \mathcal{F}_{n-1}^2}$$

or, equivalently,

$$((a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2)\alpha^2 + (\mathcal{F}_{n-1}^2 - 2\mathcal{F}_n^2 - \mathcal{F}_{n+1}^2)\alpha - \mathcal{F}_n^2 = 0. \tag{3.1}$$

For ease of exposition, we make the following change of variables: let

$$A = (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2,\ B = \mathcal{F}_{n-1}^2 - 2\mathcal{F}_n^2 - \mathcal{F}_{n+1}^2,\ C = -\mathcal{F}_n^2.$$

Since $\alpha > 0$, equation (3.1) gives

$$\alpha = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Next, if we let $x = [0, 1, \alpha] = \alpha / (\alpha + 1)$, then

$$x = \frac{2C - B + \sqrt{B^2 - 4AC}}{2(A - B + C)}.$$

The expressions $2C - B$, $B^2 - 4AC$, and $A - B + C$ may be simplified slightly by successive applications of the functional equation for $\mathcal{F}_n$. It is an algebraically complicated but straightforward task to verify that

$$x = \frac{\mathcal{F}_n}{2}\left(\frac{a(a\mathcal{F}_n + 2\mathcal{F}_{n-1}) + \sqrt{(a^2+4)(4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2)}}{4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2}\right). \tag{3.2}$$

Finally, by (2.8) and (2.9), we have

$$\mathcal{L}_n^2 = (a\mathcal{F}_n + 2\mathcal{F}_{n-1})^2 = 4\mathcal{F}_{n+1}\mathcal{F}_{n-1} + a^2\mathcal{F}_n^2,$$

and therefore, (3.2) implies

$$x = \frac{\mathcal{F}_n\left(a\mathcal{L}_n + \mathcal{L}_n\sqrt{a^2+4}\right)}{2\mathcal{L}_n^2} = \frac{\mathcal{F}_n\left(a + \sqrt{a^2+4}\right)}{2\mathcal{L}_n} = \frac{\mathcal{F}_n}{\mathcal{L}_n}\varphi_a,$$

which, by (1.4), is precisely equation (1.2) for even n.

The proof of (1.2) for n odd is similar to the even case given above. In particular, for n odd, we let

$$\alpha = \left[\overline{a^2+1, \{1, a^2\}^{(n-5)/2}, 1, a^2+1, a^2-1, 1, \{a^2, 1\}^{(n-3)/2}, a^2+3}\right].$$

Thus, in the language of matrices, we have

$$\begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix} \right\}^{(n-5)/2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2+1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2-1 & 1 \\ 1 & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} a^2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}^{(n-3)/2} \begin{pmatrix} a^2+3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

with $\alpha = r/t$. Simplifying the matrix product, as in the case for n even, reveals

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{n+1}^2 + 2\mathcal{F}_n^2 & \mathcal{F}_n^2 \\ (a^2+4)\mathcal{F}_{n-1}^2 - \mathcal{F}_{n-2}^2 & \mathcal{F}_{n-1}^2 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

Equation (1.2) for n odd now follows from the previous argument.

Equation (1.3) follows immediately from (1.2) and Theorem 11 of [4], which completes the proof.

4. THE PROOF OF THEOREM 3

We essentially adopt the argument used in the proof of Theorem 2. First, we consider the case in which n is even. Let α be the quadratic irrational defined by

$$\alpha = \left[\overline{3, \{1\}^{n-2}, L_{2n}-2, 2, \{1\}^{n-4}, 2, 1, L_{2n}-2}\right].$$

By the fundamental correspondence between matrices and continued fractions, we observe that if we express the following matrix product as

$$\begin{gathered} \left(\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} L_{2n}-2 & 1 \\ 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-4} \cdot \right. \\ \left. \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_{2n}-2 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}, \end{gathered} \tag{4.1}$$

then we have $\alpha = r/t$. Using (2.1), (2.2), (2.3), and (2.4) together with the fact that n is even, we multiply and simplify the products within the parentheses to produce

$$\begin{aligned} \begin{pmatrix} r & s \\ t & u \end{pmatrix} &= \begin{pmatrix} L_{3n} - L_{n-2} & L_n \\ F_{3n-1} + 2F_{n-2} & F_{n-1} \end{pmatrix} \begin{pmatrix} L_{3n} - F_{n-1} & L_n \\ L_{3n-2} + F_{n+3} + L_{n-3} & L_{n-2} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} k_1 &= L_{6n} - F_{4n-1} - F_{2n+1} + 1, \\ k_2 &= L_{3n} L_n, \\ k_3 &= (F_{3n-1} + 2F_{n-2})(L_{3n} - F_{n-1}) + F_{n-1}(L_{3n-2} + F_{n+3} + L_{n-3}), \\ k_4 &= F_{4n-1} + F_{2n+1} - 1. \end{aligned} \tag{4.3}$$

We note that identities (4.1) and (4.2) lead to a complicated, but useful, identity involving Fibonacci and Lucas numbers. In particular, we observe that

$$\det\left(\begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}\right) = (-1)^{2n},$$

and thus we have

$$k_1k_4 - k_2k_3 = 1. \tag{4.4}$$

By identity (4.2), we have

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} k_1\alpha + k_2 & k_1 \\ k_3\alpha + k_4 & k_2 \end{pmatrix},$$

and since $\alpha = r/t$, this implies

$$\alpha = \frac{k_1\alpha + k_2}{k_3\alpha + k_4}.$$

Therefore, $k_3\alpha^2 + (k_4 - k_1)\alpha - k_2 = 0$, hence,

$$\alpha = \frac{k_1 - k_4 + \sqrt{(k_4 - k_1)^2 + 4k_2k_3}}{2k_3}.$$

If we now let $x = [F_{n+1} - 1, \alpha]$, then

$$\begin{aligned} x &= F_{n+1} - 1 + \frac{2k_3}{k_1 - k_4 + \sqrt{(k_4 - k_1)^2 + 4k_2k_3}} \\ &= F_{n+1} - 1 + \frac{k_1 - k_4 - \sqrt{(k_4 - k_1)^2 + 4k_2k_3}}{-2k_2}. \end{aligned} \tag{4.5}$$

By (4.3), we note that $k_1 + k_4 = L_{6n}$. This, together with (4.4), reveals that

$$\begin{aligned} (k_4 - k_1)^2 + 4k_2k_3 &= (k_4 + k_1)^2 - 4k_1k_4 + 4k_2k_3 \\ &= L_{6n}^2 - 4(k_1k_4 - k_2k_3) = L_{6n}^2 - 4. \end{aligned}$$

In view of (2.5) and the fact that n is even, we may express the above as $(k_4 - k_1)^2 + 4k_2k_3 = 5F_{6n}^2$. This, along with (4.5), (2.2), (2.3), (2.4), and (2.6), yields

$$\begin{aligned} [F_{2n+1} - 1, \alpha] &= F_{2n+1} - 1 + \frac{L_{6n} - 2F_{4n-1} - 2F_{2n+1} + 2 - \sqrt{5}F_{6n}}{-2(L_{3n}L_n)} \\ &= \frac{-2F_{2n+1}L_{3n}L_n + 2L_{3n}L_n + L_{6n} - 2F_{4n-1} - 2F_{2n+1} + 2 + \sqrt{5}F_{6n}}{-2(L_{3n}L_n)} \\ &= \frac{-F_{6n} - \sqrt{5}F_{6n}}{-2L_{3n}L_n} = \frac{L_{3n}F_{3n} + L_{3n}F_{3n}\sqrt{5}}{2L_{3n}L_n} = \frac{F_{3n} + F_{3n}\sqrt{5}}{2L_n} = \frac{F_{3n}}{L_n}\varphi, \end{aligned}$$

which completes the proof for n even.

The proof for n odd is similar to the even case given above with the exception that the change of variables of (4.3) is replaced by

$$\begin{aligned} k_1 &= L_{6n} - F_{4n+1} - F_{2n-1} - 1, \\ k_2 &= L_{3n}L_n, \\ k_3 &= F_{3n+1}(L_{3n} - F_{n+1}) + F_{n+1}(L_{3n-1} + L_{n+1} + F_{n-2}), \\ k_4 &= F_{4n+1} - F_{2n-1} + 1. \end{aligned}$$

5. THE PROOF OF THEOREM 6

It is a classical result from the theory of continued fractions that α and β are equivalent numbers if and only if there exist integers a, b, c, and d so that $\alpha = (a\beta + b)/(c\beta + d)$ with $ad - bc = \pm 1$ (see [5]). Since m and n are relatively prime positive integers, we may find positive integers x and y so that $nx - my = 1$. Thus, if we let $k = 2ym$, then we also have $k = 2xn - 2$. We now define

$$a = \frac{\mathcal{L}_m \mathcal{F}_{k+2}}{\mathcal{L}_n}, \; b = c = \mathcal{F}_{k+1}, \; d = \frac{\mathcal{L}_n \mathcal{F}_k}{\mathcal{L}_m}.$$

As we remarked in the introduction, since $\mathcal{F}_{k+2} = \mathcal{F}_{(2x)n}$ and $\mathcal{F}_k = \mathcal{F}_{(2y)m}$, $\mathcal{F}_{k+2} / \mathcal{L}_n$ and $\mathcal{F}_k / \mathcal{L}_m$ are both integers. Thus, a, b, c, and d are all integers. Also, by (2.10), we note that

$$ad - bc = \mathcal{F}_k \mathcal{F}_{k+2} - \mathcal{F}_{k+1}^2 = (-1)^{k+1} = \pm 1.$$

Next, in light of (2.11), we have

$$\frac{a\left(\frac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a\right) + b}{c\left(\frac{\mathcal{L}_n}{\mathcal{L}_m}\varphi_a\right) + d} = \frac{\mathcal{L}_m}{\mathcal{L}_n}\left(\frac{\mathcal{F}_{k+2}\varphi_a + \mathcal{F}_{k+1}}{\mathcal{F}_{k+1}\varphi_a + \mathcal{F}_k}\right) = \frac{\mathcal{L}_m}{\mathcal{L}_n}\varphi_a.$$

Hence, $(\mathcal{L}_m / \mathcal{L}_n)\varphi_a$ and $(\mathcal{L}_n / \mathcal{L}_m)\varphi_a$ are equivalent numbers.

6. A RELATED CLASS OF QUADRATIC IRRATIONALS

For integers $n > 2$ and $m > 0$, we define the quadratic irrational $\mathcal{R}(n, m) = \mathcal{R}(a; n, m)$ by

$$\mathcal{R}(n, m) = \frac{a\mathcal{F}_n + \mathcal{L}_m \sqrt{a^2 + 4}}{2}.$$

It will also be useful to define the integer $N = N(n, m)$ to be $N = a\mathcal{F}_n + (a^2 + 4)\mathcal{F}_m$. We now examine the continued fraction expansion for $\mathcal{R}(n, m)$. We consider separately the case of N even and the case of N odd. As will be evident, the case of N odd is substantially more complicated that the case of N even.

Theorem 7: If N is even, then

(i) if m is even,

$$\mathcal{R}(n, m) = \left[N/2, \overline{\mathcal{F}_m, (a^2 + 4)\mathcal{F}_m}\right];$$

(ii) if m is odd and $\mathcal{F}_m > 2$,

$$\mathcal{R}(n, m) = \left[(N-2)/2, \overline{1, \mathcal{F}_m - 2, 1, (a^2 + 4)\mathcal{F}_m - 2}\right].$$

Theorem 8: If N is odd, then

(i) if m is even, $\mathcal{F}_m \equiv 0 \bmod 4$ and $\mathcal{F}_m > 4$,

$$\mathcal{R}(n, m) = \left[(N-1)/2, \overline{1, 1, (\mathcal{F}_m - 4)/4, 1, 1, (a^2 + 4)\mathcal{F}_m - 1}\right];$$

(ii) if m is even, $\mathcal{F}_m \equiv 1 \bmod 4$ and $\mathcal{F}_m > 5$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\vec{\mathcal{W}}_1, 4\mathcal{F}_m, \overleftarrow{\mathcal{W}}_1, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_1 = \left(1, 1, (\mathcal{F}_m - 5)/4, 1, 3, ((a^2+4)\mathcal{F}_m - 1)/4\right)$;

(iii) if m is even, $\mathcal{F}_m \equiv 3 \bmod 4$ and $\mathcal{F}_m > 3$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\vec{\mathcal{W}}_2, 4\mathcal{F}_m, \overleftarrow{\mathcal{W}}_2, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_2 = \left(1, 1, (\mathcal{F}_m - 3)/4, 3, 1, ((a^2+4)\mathcal{F}_m - 3)/4\right)$;

(iv) if m is odd, $\mathcal{F}_m \equiv 1 \bmod 4$ and $F_m > 5$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\vec{\mathcal{W}}_3, \mathcal{F}_m - 1, \overleftarrow{\mathcal{W}}_3, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_3 = \left(2, (\mathcal{F}_m - 5)/4, 1, 2, 1, ((a^2+4)\mathcal{F}_m - 5)/4, 2\right)$;

(v) if m is odd, $\mathcal{F}_m \equiv 2 \bmod 4$ and $F_m > 6$,

$$\mathcal{R}(n,m) = \left[(N-1)/2, \overline{\vec{\mathcal{W}}_4, 4(a^2+4)\mathcal{F}_m - 2, \overleftarrow{\mathcal{W}}_4, (a^2+4)\mathcal{F}_m - 1}\right],$$

where $\mathcal{W}_4 = (2, (\mathcal{F}_m - 6)/4, 1)$.

Since the proof of Theorem 8 involves the same ideas as the proof of Theorem 7, we include only the (less complicated) proof of Theorem 7. Before proceeding with the proof of Theorem 7, we make three remarks.

First, it may appear that Theorem 8 is not complete in the sense that three cases seem to be missing; in particular, the cases: m even, $\mathcal{F}_m \equiv 2 \bmod 4$; m odd, $\mathcal{F}_m \equiv 0 \bmod 4$; m odd, $\mathcal{F}_m \equiv 3 \bmod 4$. It is a straightforward calculation to verify that none of these cases can occur when N is odd. For example, one has that $\mathcal{F}_m \equiv 2 \bmod 4$ only if either $a \equiv 1 \bmod 4$ and $m \equiv 3 \bmod 6$ or $a \equiv 3 \bmod 4$ and $m \equiv 3 \bmod 6$ or $a \equiv 2 \bmod 4$ and $m \equiv 2 \bmod 4$. In the first two cases, m is odd, and in the third case a is even; thus, N must be even. So if $\mathcal{F}_m \equiv 2 \bmod 4$, then we cannot have both m even and N odd. Similarly, the other two remaining cases may be shown not to occur. Therefore, Theorem 8 gives the complete situation for odd N. Our second remark involves the numbers $((a^2+4)\mathcal{F}_m - 1)/4$, $((a^2+4)\mathcal{F}_m - 3)/4$, and $((a^2+4)\mathcal{F}_m - 5)/4$ occurring in cases (ii), (iii), and (iv), respectively. Of course, we must require that these be integers. It is easy to see that each is an integer in the appropriate case if and only if a is odd. However, again, if a were even, then N would be even and Theorem 7 would apply. Hence, if N is odd, then a is also odd; there-fore, the three numbers above are indeed integers as required. Third, we note that the period length for $\mathcal{R}(n,m)$ is either 2, 4, 6, 8, 14, or 16. This proves an observation made by Long [2] that the period is always even, but it also shows that the period length is, in fact, a bounded func-tion of a, n, and m which Long believed not to be the case.

Proof of Theorem 7: We consider first the case of m even and let

$$\alpha = \left[N/2, \overline{\mathcal{F}_m, (a^2+4)\mathcal{F}_m}\right].$$

We now examine the corresponding matrix product:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} N/2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{F}_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (a^2+4)\mathcal{F}_m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (a^2+4)\mathcal{F}_m + \alpha - N/2 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows that $\alpha = r/t$, in particular,

$$\alpha = \frac{(N\mathcal{F}_m/2+1)((a^2+4)\mathcal{F}_m + \alpha - N/2) + N/2}{\mathcal{F}_m((a^2+4)\mathcal{F}_m + \alpha - N/2) + 1}.$$

Thus, we have

$$\mathcal{F}_m\alpha^2 + (\mathcal{F}_m((a^2+4)\mathcal{F}_m - N/2) - N\mathcal{F}_m/2)\alpha \\ + (N/2)^2\mathcal{F}_m - (a^2+4)\mathcal{F}_m - N(a^2+4)\mathcal{F}_m^2/2 = 0$$

which, together with our definition of N yields

$$\alpha = \frac{a\mathcal{F}_n + \sqrt{((a^2+4)\mathcal{F}_m^2+4)(a^2+4)}}{2}.$$

As m is even, identity (2.12) becomes $(a^2+4)\mathcal{F}_m^2 + 4 = \mathcal{L}_m^2$; hence, $\alpha = \mathcal{R}(n,m)$.

If m is odd, we again let

$$\alpha = \left[(N-2)/2, \overline{1, \mathcal{F}_m - 2, 1, (a^2+4)\mathcal{F}_m - 2}\right],$$

and proceed in a similar manner to deduce

$$\alpha = \frac{N - (a^2+4)\mathcal{F}_m + \sqrt{((a^2+4)\mathcal{F}_m^2 - 4)(a^2+4)}}{2}.$$

In view of identity (2.12) with m odd, together with the definition of N, we have $\alpha = \mathcal{R}(n,m)$, which completes the proof.

As a consequence of the two previous theorems and a result of Long [2], we are able to deduce immediately the continued fraction expansion for numbers of the form

$$\mathcal{S}(n,m) = \frac{a\,\mathcal{L}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}.$$

Long proved (Theorem 8, [2]) that the continued fraction expansions of $\mathcal{R}(n,m)$ and $\mathcal{S}(n,m)$ are identical after the first partial quotient. In view of the two theorems of this section, it appears clear that one may explicitly express the continued fraction expansion for

$$\frac{a\mathcal{F}_n + \mathcal{L}_m\sqrt{a^2+4}}{2}$$

and, thus, by Theorem 9 of [2], the expansion for

$$\frac{a\mathcal{F}_n + \mathcal{F}_m\sqrt{a^2+4}}{2}.$$

It seems very reasonable to conjecture that these period lengths will again be even and bounded.

ACKNOWLEDGMENT

Part of the work herein resulted from the Number Theory Group [1] of the Williams College SMALL Undergraduate Research Project, Summer 1993. The Group members were Christopher French, Christopher Kollett, Michael Pelsmajer, Jeffrey Vander Clute, and Aaron Williams, with Edward Burger as the faculty advisor. Research was supported, in part, by a grant from The National Science Foundation Research Experiences for Undergraduates Program.

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AMS Classification Numbers: 11A55, 11J70

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