A FIBONACCI MODEL OF INFECTIOUS DISEASE

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1. INTRODUCTION

The Fibonacci rabbit population model is often regarded as one of the first studies of population growth using mathematics. Later, an analytic model of population dynamics was introduced by Voltera (Deakin & McElwain [2]). Systematic epidemic modeling in age-structured populations was first carried out in this century by Hoppensteadt [6].

Dubeau [3], in revisiting the Fibonacci rabbit growth model, has developed an approach that can be applied to population dynamics and epidemiology where censoring occurs either by inability to procreate or by death. It is the purpose of this note to apply Dubeau's method to Fibonacci's model of infectious diseases which was developed by Makhmudov [9] and to combine it with the approach of Shannon et al. [12] who attempted to refine the work of Makhmudov.

2. THE MODEL

Following Makhmudov [9], three epidemiological stages in the process of spreading infectious diseases are postulated:

(i) an initial (incubation) stage of r periods (periods 0, 1, 2, ..., $r-1$) during which those who are ill with the disease do not affect others,
(ii) a mature (infectious) stage of t periods (periods $r, r+1, \ldots, r+t-1$) when each person infects s healthy people, and
(iii) a removal stage of m periods (periods $r+t, \ldots, r+t+m-1$) when those who have been infected are no longer infectious.

An example might be the common cold which, on average, takes about two days to develop ($r=2$), a person is then infectious for about three days ($t=3$), and the symptoms persist for about seven days ($r+t+m=7$, hence $m=2$). In general, s is variable, but we shall treat it as a constant in the absence of other information. For background material on the structure of general epidemic models, the reader is referred to Billiard & Zhen Zhao [1].

In terms of a modification to Fibonacci's rabbit problem, these correspond in turn to

(i) the infancy stage,
(ii) the reproductive stage, and
(iii) the post reproductive stage,

respectively, and instead of infectives we have male-female pairs of rabbits. For the original Fibonacci model, we take $r=2$, $s=1$, and $t=m=+\infty$.

2.1 A Direct Approach

Following Dubeau [3], let

u_n be the total number of disease carriers at the n^{th} period, and

v_n^i be the number of i-period old disease carriers at the n^{th} period.

More precisely, v_n^i represents the disease carriers in the

(i) initial stage for $i = 0, \ldots, r-1$,
(ii) mature stage for $i = r, \ldots, r+t-1$,
(iii) removal reproductive stage for $i = r+t, \ldots, r+t+m-1$,

and for $i = r+t+m, \ldots$, v_n^i represents the disease carriers who have been infected in the past but have already recovered.

It will be convenient to define u_n and v_n^i for all $n \in Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ and $i \in N = \{0, 1, 2, 3, \ldots\}$. We consider the following initial conditions on v_n^i:

$$v_n^i = \begin{cases} 0 & \text{for } \begin{cases} n < 0 \text{ and } i = 0, 1, 2, \ldots, \\ n = 0 \text{ and } i = 1, 2, 3, \ldots, \end{cases} \\ 1 \ (\text{or } v_0^0) & \text{for } n = 0 \text{ and } i = 0. \end{cases}$$

As a consequence, for any $n \in Z$,

$$v_n^i = v_{n-1}^{i-1} \quad \text{for } i > 0,$$

and

$$v_n^0 = \begin{cases} 1 & \text{for } n = 0, \\ s\{v_n^r + \cdots + v_n^{r+t-1}\} & \text{for } n \neq 0. \end{cases}$$

We obtain

$$v_n^0 = v_{n-1}^0 + s\{v_{n-r}^0 - v_{n-r-t}^0\} \tag{1}$$

for $n > 1$. From the definition we have, for any $n \in Z$,

$$u_n = \sum_{i=0}^{r+t+m-1} v_n^i.$$

It follows that

$$u_n = \begin{cases} 0 & \text{for } n < 0, \\ u_0 + s\sum_{k=r}^{r+t-1} u_{n-k} & \text{for } n = 0, \ldots, r+t+m-1, \\ s\sum_{k=r}^{r+t-1} u_{n-k} & \text{for } n \geq r+t+m, \end{cases} \tag{2}$$

or

$$u_n = u_{n-1} - \delta_{n, r+t+m} u_0 + s\{u_{n-r} - u_{n-r-t}\} \tag{3}$$

for $n \geq 1$, where $\delta_{i,j} = 0$ if $i \neq j$, or 1 if $i = j$.

Example 1: Table 1 contains values of v_n^j and u_n for $r = 2, t = 3, m = 2,$ and $s = 1$.

TABLE 1. $r = 2, t = 3, m = 2,$ and $s = 1$

	u_n							
$n \backslash i$	0	1	2	3	4	5	6	u_n
0	1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1
2	1	0	1	0	0	0	0	2
3	1	1	0	1	0	0	0	3
4	2	1	1	0	1	0	0	5
5	2	2	1	1	0	1	0	7
6	4	2	2	1	1	0	1	11
7	5	4	2	2	1	1	0	15
8	8	5	4	2	2	1	1	23
9	11	8	5	4	2	2	1	33
10	17	11	8	5	4	2	2	49
11	24	17	11	8	5	4	2	71
12	36	24	17	11	8	5	4	105
13	52	36	24	17	11	8	5	153
14	77	52	36	24	17	11	8	225
15	112	77	52	36	24	17	11	329
16	165	112	77	52	36	24	17	483
17	241	165	112	77	52	36	24	707
18	354	241	165	112	77	52	36	1037
19	518	354	241	165	112	77	52	1519
20	760	518	354	241	165	112	77	2227

Remark—The Effect of m: Let $\{u_n\}_{n=0}^{+\infty}$ and $\{\tilde{u}_n\}_{n=0}^{+\infty}$ be the sequences generated with m and $m+1$ for the same values of r, t, and s. From (3) we have, for $n \geq 1$,

$$u_n = u_{n-1} - \delta_{n,\,r+t+m} u_0 + s\{u_{n-r} - u_{n-r-t}\}$$

and

$$\tilde{u}_n = \tilde{u}_{n-1} - \delta_{n,\,r+t+m+1} u_0 + s\{\tilde{u}_{n-r} - \tilde{u}_{n-r-t}\}.$$

Let $\Delta_n u = u_n - \tilde{u}_n$, then from (3)

$$\Delta_n u = \Delta_{n-1} u + (\delta_{n,\, r+t+m} - \delta_{n,r+t+m+1}) u_0 + s\{\Delta_{n-r} u - \Delta_{n-r-t} u\}.$$

It follows from (1) that $\Delta_{r+t+m+n} u = v_n^0$ for $n \geq 0$.

2.2 A Generating Function Approach

Following Weland [13], Hoggatt [4], Hoggatt & Lind [5], and Parberry [11], we can use the generating function method to obtain the recurrence relation (1), (2), or (3).

Let us define the generating function for the sequence $\{u_n\}_{n=0}^{+\infty}$:

$$U(x) = \sum_{n=0}^{+\infty} u_n x^n.$$

The function $U(x)$ can be expressed in terms of (i) the generating function of the infectious process (a polynomial)

$$B(x) = \sum_{n=0}^{+\infty} b_n x^n \quad (b_0 = 0),$$

where b_n indicates the number of infected healthy people by an n-period old disease carrier, and (ii) the "total recovering" polynomial $D(x) = x^{r+t+m}$.

Let

$$V(x) = \sum_{n=0}^{+\infty} v_n^0 x^n$$

be the generating function associated to the sequence $\{v_n^0\}_{n=0}^{+\infty}$, where

$$v_0^0 = 1,$$
$$v_1^0 = b_0 v_1^0 + b_1 v_0^0,$$
$$v_2^0 = b_0 v_2^0 + b_1 v_1^0 + b_2 v_0^0$$
etc.,

and, in general,

$$v_n^0 = \sum_{j=0}^{n} b_j v_{n-j}^0$$

for $n \geq 1$. It follows that

$$V(x) = \frac{1}{1 - B(x)}.$$

Let u_n^* be the number of disease carriers at the n^{th} period, assuming no recovery, and

$$U^*(x) = \sum_{n=0}^{+\infty} u_n^* x^n.$$

Then

$$u_n^* = \sum_{j=0}^{n} v_j^0$$

and we obtain

$$U^*(x) = \frac{1}{(1-x)}V(x) = \frac{1}{(1-x)(1-B(x))}.$$

If we now allow for recoveries, since each disease carrier recovers $r+t+m$ periods after its infection, the number h_n of recovering people at the n^{th} period is given by $h_n = v^0_{n-(r+t+m)}$. Therefore,

$$H(x) = \sum_{n=0}^{+\infty} h_n x^n = D(x)V(x) = \frac{D(x)}{1-B(x)}.$$

Let r_n be the total number of people who recovered up to the n^{th} period, then

$$r_n = \sum_{j=0}^{n} h_j$$

and

$$R(x) = \sum_{n=0}^{+\infty} r_n x^n = \frac{1}{1-x}H(x) = \frac{D(x)}{(1-x)(1-B(x))}.$$

Now, $u_n = u_n^* - r_n$ $(n \geq 0)$, so that

$$U(x) = U^*(x) - R(x) = \frac{1-D(x)}{(1-x)(1-B(x))}.$$

From the model, we have

$$B(x) = s\sum_{n=r}^{r+t-1} x^n, \qquad D(x) = x^{r+t+m},$$

and

$$U(x) = \frac{1-x^{r+t+m}}{1-x-sx^r+sx^{r+t}}.$$

It follows that $u_0 = 1$ and, for $n \geq 1$, $u_n = u_{n-1} - \delta_{n,r+t+m}u_0 + s\{u_{n-r} - u_{n-(r+t)}\}$. Moreover, since

$$V(x) = \frac{1}{1-s\sum_{n=r}^{r+t-1} x^n},$$

it follows that $v_0^0 = 1$, and $v_n^0 = s\{v_{n-r}^0 + \cdots + v_{n-(r+t-1)}^0\}$ for $n \geq 1$.

2.3 A Matrix Approach

Following Klarner [8], let us consider the sequence of $(r+t+m)$-vectors $\{v_n\}_{n=0}^{+\infty}$:

$$v_n = [v_n^0, v_n^1, \ldots, v_n^{r+t+m-1}] \quad (n = 0, 1, 2, \ldots).$$

They are related by the equation

$$v_{n+1} = v_n F = \cdots = v_0 F^{n+1}, \tag{4}$$

where $v_0 = [1, 0, \ldots, 0]$ and $F = (f_{ij})$ is a square matrix of order $r+t+m$ with entries f_{ij} $(i = 0, \ldots, r+t+m-1;\ j = 0, \ldots, r+t+m-1)$ such that

$$v_{n+1}^{j} = \sum_{i=0}^{r+t+m-1} v_n^i f_{ij}.$$

For our problem,

$$f_{i0} = \begin{cases} s & \text{for } i = r, \ldots, r+r-1, \\ 0 & \text{elsewhere,} \end{cases}$$

and for $j = 1, \ldots, r+t+m-1$,

$$f_{ij} = \begin{cases} 1 & \text{if } i = j-1, \\ 0 & \text{elsewhere.} \end{cases}$$

The characteristic polynomial of F is $\det(xI - F) = x^{r+t+m} - s(x^{t+m} + \cdots + x^{1+m}) = c_F(x)$. From the Cayley-Hamilton theorem, the matrix F satisfies its characteristic equation, and we have $c_F(F) = 0$. Hence, $F^n c_F(F) = 0$ for any $n \geq 0$. It follows that $F^n - s(F^{n-r} + \cdots + F^{n-(r+t-1)}) = 0$ for $n \geq r+t+m$. Finally, from (4), we have

$$v_n - s(v_{n-r} + \cdots + v_{n-(r+t-1)}) = 0$$

and, since $u_n = v_n 1$, where $1 = [1, \ldots, 1]^T$,

$$u_n - s(u_{n-r} + \cdots + u_{n-(r+t-1)}) = 0$$

for $n \geq r+t+m$.

3. A RELATED ARRAY

Let $w_n^i(k)$ be the number of i-period old disease carriers of the k^{th} generation at the n^{th} period, and $w_n(k)$ be the number of disease carriers of the k^{th} generation at the n^{th} period. We have

$$w_n(k) = \sum_{i=0}^{r+t+m-1} w_n^i(k)$$

and, for $i \geq r+t+m$, $w_n^i(k)$ indicates the number of people of the k^{th} generation at the n^{th} period infected i periods ago and who have already recovered.

We also have

$$w_n^i(k) = 0 \qquad \text{for } n < 0 \text{ or } i < 0,$$
$$w_n^i(k) = \delta_{ni} \qquad \text{for } n \geq 0 \text{ and } i \geq 0,$$

and, for $k \geq 1, n \geq 0$, and $i \geq 0$,

$$w_n^i(k) = \begin{cases} s\{w_n^r(k-1) + \cdots + w_n^{r+t-1}(k-1)\} & \text{for } i = 0, \\ w_{n-1}^{i-1}(k) & \text{for } i \geq 1. \end{cases}$$

We can deduce that $w_n^0(k) \neq 0$ for $kr \leq n \leq k(r+t-1)$, $k = 0, 1, 2, \ldots$. It follows that $w_n(k) \neq 0$ only for $k \geq 0$ and $n \geq 0$ such that $kr = n_L(k) \leq n \leq n_U(k) = (r+t+m-1) + k(r+t-1)$. Then, for a given n, let

$$k_L(n) = \min\left\{k \in N \mid k \geq 0 \text{ and } k \geq \frac{n-(r+t+m-1)}{r+t-1}\right\},$$

$$k_U(n) = \max\left\{k \in N \mid k \leq \frac{n}{r}\right\};$$

hence, $w_n(k) = 0$ for $k < k_L(n)$ and $k > k_U(n)$. To relate the $w_n(k)$ and $w_n^i(k)$ to the u_n and v_n^i, we have

$$u_n = \sum_{k=0}^{+\infty} w_n(k) = \sum_{k=k_L(n)}^{k_U(n)} w_n(k),$$

$$v_n^i = \sum_{k=0}^{+\infty} w_n^i(k) = \sum_{k=0}^{k_U(n)} w_n^i(k).$$

Also, for $k \geq 1$,

$$\begin{aligned} w_n(k) &= \sum_{i=0}^{r+t+m-1} w_n^i(k) = \sum_{i=0}^{r+t+m-1} w_{n-i}^0(k) \\ &= \sum_{i=0}^{r+t+m-1} s \sum_{\ell=r}^{r+t-1} w_{n-i}^{\ell}(k-1) = s \sum_{\ell=r}^{r+t-1} \sum_{i=0}^{r+t+m-1} w_{n-i}^{\ell}(k-1) \\ &= s \sum_{\ell=r}^{r+t-1} \sum_{i=0}^{r+t+m-1} w_{n-\ell}^{i}(k-1) = s \sum_{\ell=r}^{r+t-1} w_{n-\ell}(k-1). \end{aligned}$$

As a consequence, using the generating function, we have

$$\begin{aligned} G_k(x) &= \sum_{n=0}^{+\infty} w_n(k)x^n = s\sum_{n=0}^{+\infty}\left(\sum_{\ell=r}^{r+t-1} w_{n-\ell}(k-1)\right)x^n \\ &= s\sum_{n=0}^{+\infty}\left(\sum_{\ell=n-(r+t-1)}^{n-r} w_{\ell}(k-1)\right)x^n = s(x^r + \cdots + x^{r+t-1})\sum_{n=0}^{+\infty} w_n(k-1)x^n \\ &= s(x^r + \cdots + x^{r+t-1})G_{k-1}(x). \end{aligned}$$

Also,

$$G_0(x) = \sum_{n=0}^{+\infty} w_n(0)x^n = 1 + x + \cdots + x^{r+t+m-1},$$

thus

$$G_k(x) = s^k\left[x^r(1 + x + \cdots + x^{t-1})\right]^k G_0(x)$$

and

$$G_k(1) = (st)^k(r+t+m) = \sum_{n=n_L(k)}^{u_U(k)} w_n(k).$$

Example 2: Table 2 illustrates the values of $w_n^i(k)$ and $w_n(k)$ for $r = 2, t = 3, m = 2$, and $s = 1$.

TABLE 2. $r = 2, t = 3, m = 2$, and $s = 1, n_L(k) = 2k, n_U(k) = 4k + 6$,
$k_L(n) = \min\left\{k \in N \mid k \geq 0 \text{ and } k \geq \frac{n-6}{4}\right\}$, $k_U(n) = \max\left\{k \in N \mid k \leq \frac{n}{2}\right\}$

		$w_n^i(k)$							
		i							
n	$k_L(n) \leq k \leq k_U(n)$	0	1	2	3	4	5	6	$w_n(k)$
0	$k_L(0) = 0 = k_U(0)$	1	0	0	0	0	0	0	1
1	$k_L(1) = 0 = k_U(1)$	0	1	0	0	0	0	0	1
2	$k_L(2) = 0$	0	0	1	0	0	0	0	1
	$1 = k_U(2)$	1	0	0	0	0	0	0	1
3	$k_L(3) = 0$	0	0	0	1	0	0	0	1
	$1 = k_U(3)$	1	1	0	0	0	0	0	2
4	$k_L(4) = 0$	0	0	0	0	1	0	0	1
	1	1	1	1	0	0	0	0	3
	$2 = k_U(4)$	1	0	0	0	0	0	0	1
5	$k_L(5) = 0$	0	0	0	0	0	1	0	1
	1	0	1	1	1	0	0	0	3
	$2 = k_U(5)$	2	1	0	0	0	0	0	3
6	$k_L(6) = 0$	0	0	0	0	0	0	1	1
	1	0	0	1	1	1	0	0	3
	2	3	2	1	0	0	0	0	6
	$3 = k_U(6)$	1	0	0	0	0	0	0	1
7	$k_L(7) = 1$	0	0	0	1	1	1	0	3
	2	2	3	2	1	0	0	0	8
	$3 = k_U(7)$	3	1	0	0	0	0	0	4
8	$k_L(8) = 1$	0	0	0	0	1	1	1	3
	2	1	2	3	2	1	0	0	9
	3	6	3	1	0	0	0	0	10
	$4 = k_U(8)$	1	0	0	0	0	0	0	1
9	$k_L(9) = 1$	0	0	0	0	0	1	1	2
	2	0	1	2	3	2	1	0	9
	3	7	6	3	1	0	0	0	17
	$4 = k_U(9)$	4	1	0	0	0	0	0	5

4. LIMIT OF RATIOS u_{n+1} / u_n

We consider the linear difference equation (2) of order $r + t - 1$:

$$u_n = s \sum_{k=r}^{r+t-1} u_{n-k} \quad (n \geq r + t + m).$$

The sequence $\{u_n\}_{n=r+t+m}^{+\infty}$ is completely defined if we assume that the values u_{m+1}, u_{m+2}, ..., $u_{r+t+m-1}$ are known.

For our model (2) or (3), we observe that the finite sequence $\{u_n\}_{n=0}^{r+t+m-1}$ is a sequence of non-decreasing integers with $u_0 = 1$ (or any initial value $u_0 > 0$).

We consider two cases for the analysis of the ratios u_{n+1}/u_n: the case $t = 1$ and the case $t > 1$.

4.1 The Case $t = 1$

We have $u_n = su_{n-r}$ $(n \geq r+m+1)$. It follows that

$$\frac{u_{n+r+1}}{u_{n+r}} = \frac{u_{n+1}}{u_n} \quad (n \geq m+1)$$

and the sequence of ratios u_{n+1}/u_n is a sequence of length r repeated infinitely many times. It is completely characterized by the finite sequence

$$\frac{u_{n+1}}{u_n} \quad \text{for } n = m+1, \ldots, m+r.$$

Using (2), for the initial value $u_0 = 1$, we have

$$u_n = \sum_{i=0}^{k} s^i \quad \text{for} \begin{cases} n \leq r+m, \text{ and} \\ kr \leq n < (k+1)r \end{cases}$$

and

$$u_n = \sum_{i=0}^{\left\lfloor \frac{n}{r} \right\rfloor} s^i \quad \text{for } n = 0, \ldots, r+m.$$

Let

$$\rho_U = \left\lfloor \frac{r+m}{r} \right\rfloor \quad \text{and} \quad \rho_L = \left\lfloor \frac{1+m}{r} \right\rfloor,$$

then $\rho_U = \rho_L$ or $\rho_L + 1$. Hence, the sequence $\left\{\frac{u_{n+1}}{u_n}\right\}_{n=m+1}^{m+r}$ is such that

$$\frac{u_{n+1}}{u_n} = \begin{cases} 1 & r-2 \text{ times}, \\ s\sum_{i=0}^{\rho_L} s^i \Big/ \sum_{i=0}^{\rho_U} s^i & 1 \text{ time}, \\ \sum_{i=0}^{\rho_U} s^i \Big/ \sum_{i=0}^{\rho_L} s^i & 1 \text{ time}. \end{cases} \tag{6}$$

It can be shown that the set

$$\left\{ \frac{u_{n+1}}{u_n} \,\middle|\, n = m+1, \ldots, m+r \right\}$$

converges to the set $\{1, s\}$ when m goes to $+\infty$.

Example 3: Table 3 contains the values of (6) for $r = 4$, $t = 1$, and $s = 2$.

TABLE 3. $r = 4$, $t = 1$, and $s = 2$

$\frac{u_{m+i+1}}{u_{m+i}}$	m								
i	1	2	3	4	5	6	7	8	9
1	1	3	1	1	1	7/3	1	1	1
2	3	1	1	1	7/3	1	1	1	15/7
3	1	1	1	7/3	1	1	1	15/7	1
4	2/3	2/3	2	6/7	6/7	6/7	2	14/15	14/15

4.2 The Case $t > 1$

Let $K = r + t - 1$. The linear difference equation (5) is equivalent to the following linear difference equation of order K,

$$u_{n+K} = s\sum_{k=0}^{t-1} u_{n+k} \quad (n \geq 0)$$

if the sequence $\{u_n\}_{n=0}^{K-1}$ for (7) corresponds to the sequence $\{u_n\}_{n=m+1}^{m+K}$ for (5). Hence, the limit of u_{n+1}/u_n is the same for both equations.

Let us recall some definitions and results about linear difference equations of the form

$$u_{n+K} - b_1 u_{n+K-1} - \cdots - b_{K-1} u_{n+1} - b_K u_n = 0 \quad (n \geq 0). \tag{8}$$

Definitions:

(a) The polynomial $\varphi(\lambda) = \lambda^K - b_1\lambda^{K-1} - \cdots - b_K$ is called the characteristic polynomial for (8).

(b) The equation $\varphi(\lambda) = 0$ is the characteristic equation for (8).

(c) The solutions $\lambda_1, \ldots, \lambda_\ell$ of the characteristic equation are the characteristic roots.

The first result is a standard result about the general solution of (8).

Theorem 1: Suppose (8) has characteristic roots $\lambda_1, \ldots, \lambda_k$ with multiplicities $j_1, \ldots, j_k$, respectively. Then (8) has n independent solutions $n^j\lambda_\ell^n$, $j = 0, \ldots, j_\ell - 1$; $\ell = 1, \ldots, k$. Moreover, any solution of (8) is of the form

$$u_n = \sum_{\ell=1}^{k}\sum_{j=0}^{j_\ell-1} \beta_{\ell,j} n^j \lambda_\ell^n \quad (n \geq 0),$$

where the $\beta_{\ell,j}$ are obtained from the values of u_n for $n = 0, \ldots, K-1$.

Proof: See, for example, Kelley & Peterson [7]. □

The next two results depend on the form of (8).

Theorem 2: Assume the b_i are nonnegative in (8).

(a) If at least one b_i is strictly positive, then (8) has a unique simple characteristic root $\sigma > 0$ and all other characteristic roots of (8) have moduli not greater than σ.

(b) If the indices of the b_i that are strictly positive have the common greatest divisor 1, then (8) has a unique simple characteristic root $\sigma > 0$, and the moduli of all other characteristic roots of (8) is strictly less than σ.

Proof: See Ostrowski [10, pp. 91-92]. □

Theorem 3: If in (8) the b_i are nonnegative and $\{u_n\}_{n=0}^{+\infty}$ is a sequence satisfying (8) such that u_0, u_1, ..., u_{K-1} are strictly positive, then we have $u_n \geq \alpha\sigma^n$ $(n \geq 0)$, where $\alpha > 0$ is given by

$$\alpha = \min\left\{\frac{u_n}{\sigma^n} \middle| n = 0, \ldots, K-1\right\}.$$

Proof: See Ostrowski [10, p. 93]. □

Since

$$b_i = \begin{cases} 0 & \text{for } i = 1, \ldots, r-1, \\ s & \text{for } i = r, \ldots, r+t-1, \end{cases}$$

and the common greatest divisor of $r, \ldots, r+t-1$ is 1 for $t > 1$, it follows from Theorem 2(b) that (7) as a unique simple characteristic root $\sigma > 0$ and the moduli of all other characteristic roots are less than σ.

Let $\lambda_1, \ldots, \lambda_k$ and σ be the characteristic roots of (7), then, from Theorem 1,

$$u_n = \beta\sigma^n + \sum_{\ell=1}^{k} \sum_{j=0}^{j_\ell - 1} \beta_{\ell,j} n^j \lambda_\ell^n.$$

Moreover, since $u_0 \geq 1$ and $\{u_n\}_{n=0}^{K-1}$ is a nondecreasing sequence, we obtain, from Theorem 3, $u_n \geq \alpha\sigma^n$ for

$$\alpha = \min\left\{\frac{u_n}{\sigma^n} \middle| n = 0, \ldots, K-1\right\}.$$

It follows that

$$\alpha \leq \frac{u_n}{\sigma^n} = \beta + \sum_{\ell=1}^{k} \sum_{j=0}^{j_\ell - 1} \beta_{\ell,j} n^j \left(\frac{\lambda_\ell}{\sigma}\right)^n,$$

and taking the limit on both sides we have $\lim_{n\to+\infty} u_n / \sigma^n = \beta \geq \alpha > 0$ as a consequence of the following lemma.

Lemma: If $|\rho| < 1$, then $\lim_{n\to+\infty} n^\alpha \rho^n = 0$ for any $\alpha = 0, 1, 2, \ldots$. □

Finally,

$$\frac{u_{n+1}}{u_n} = \sigma \frac{u_{n+1} / \sigma^{n+1}}{u_n / \sigma^n}$$

and we obtain $\lim_{n\to+\infty} u_{n+1}/u_n = \sigma$, where σ is the unique positive root of

$$\varphi(x) = x^{r+t-1} - s\sum_{i=0}^{t-1} x^i \quad (t > 1).$$

5. A MORE REALISTIC MODEL

In any real population, the epidemiological status of members is as follows: (i) susceptibles, (ii) infected, and (iii) resistants. Thus, there is not an unlimited supply of susceptibles.

Let

N be the total population,
S_n be the number of susceptibles at the n^{th} period,
U_n be the number of infected and carriers at the n^{th} period, and
R_n be the number of resistants at the n^{th} period.

Then $N = S_n + U_n + R_n$, and the initial conditions are $S_0 = N-1$, $U_0 = 1$, and $R_0 = 0$. Using the notation of Section 2, we have

$$U_n = \sum_{i=0}^{r+t+m-1} v_n^i$$

$$R_n = \sum_{i=r+t+m}^{+\infty} v_n^i = \sum_{i=r+t+m}^{+\infty} v_{n-i}^0 = \begin{cases} 0 & \text{if } n < r+t+m, \\ \sum_{i=r+t+m}^{n} v_{n-i}^0 & \text{if } n \geq r+t+m, \end{cases}$$

and

$$S_n = N - U_n - R_n.$$

However, the number of susceptibles is limited, so

$$v_n^0 = \min\left\{S_{n-1}, s\sum_{i=r}^{r+t-1} v_n^i\right\}$$

and

$$S_n = S_{n-1} - v_n^0,$$

$$U_n = U_{n-1} + v_n^0 - v_n^{r+t+m},$$

$$R_n = R_{n-1} + v_n^{r+t+m}.$$

For R_n we have

$$\begin{aligned} R_n &= R_{n-(r+t+m)} + \sum_{i=0}^{r+t+m-1} v_{n-i}^{r+t+m} \\ &= R_{n-(r+t+m)} + \sum_{i=0}^{r+t+m-1} v_{n-(r+t+m)}^i \\ &= R_{n-(r+t+m)} + U_{n-(r+t+m)} = N - S_{n-(r+t+m)}. \end{aligned}$$

It follows that

$$S_n - S_{n-(r+t+m)} + U_n = 0$$

or

$$U_n = S_{n-(r+t+m)} - S_n$$

and, if $S_n = 0$, then $U_n = S_{n-(r+t+m)}$.

Example 4: Table 4 illustrates this model for $r = 2, t = 3, m = 2, s = 1,$ and $N = 200$.

TABLE 4. $r = 2, t = 3, m = 2, s = 1,$ and $N = 200$

	v_n^i									
n \ i	0	1	2	3	4	5	6	U_n	R_n	S_n
0	1	0	0	0	0	0	0	1	0	199
1	0	1	0	0	0	0	0	1	0	199
2	1	0	1	0	0	0	0	2	0	198
3	1	1	0	1	0	0	0	3	0	197
4	2	1	1	0	1	0	0	5	0	195
5	2	2	1	1	0	1	0	7	0	193
6	4	2	2	1	1	0	1	11	0	189
7	5	4	2	2	1	1	0	15	1	184
8	8	5	4	2	2	1	1	23	1	176
9	11	8	5	4	2	2	1	33	2	165
10	17	11	8	5	4	2	2	49	3	148
11	24	17	11	8	5	4	2	71	5	124
12	36	24	17	11	8	5	4	105	7	88
13	52	36	24	17	11	8	5	153	11	36
14	36	52	36	24	17	11	8	184	16	0
15	0	36	52	36	24	17	11	176	24	0
16	0	0	36	52	36	24	17	165	35	0
17	0	0	0	36	52	36	24	148	52	0
18	0	0	0	0	36	52	36	124	76	0
19	0	0	0	0	0	36	52	88	112	0
20	0	0	0	0	0	0	36	36	164	0
21	0	0	0	0	0	0	0	0	200	0

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