ON THE UNIQUENESS OF REDUCED PHI-PARTITIONS

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1. PRELIMINARIES

For any positive integer k, let $P_k = k^{\text{th}}$ prime and define $s_k = \prod_{j=1}^{k-1} P_j$ A positive integer n is *simple* if $n = s_k$ for some positive integer k.

A ϕ-*partition* of n is a partition $n = d_1 + \cdots + d_i$, where i and $d_1, \ldots, d_i$ are positive integers, that satisfies the condition $\phi(n) = \phi(d_1) + \cdots + \phi(d_i)$, where ϕ is the Euler phi-function. In [1], Jones shows that the simple integers s_k have only the trivial ϕ-partition $s_k = s_k$, and so define a ϕ-partition of a positive integer n to be *reduced* if all the summands are simple.

Writing a partition as $\sum_{j=1}^{i} b_j * d_1 + \cdots + b_i * d_i$, means that d_j occurs b_j times in the partition for $j = 1, \ldots, i$.

Every positive integer n has a unique partition $\sum_{j=1}^{i} c_j * s_j$ satisfying the condition that $0 \le c_j < P_j$ for $j \ge 1$. This partition is a special type of *Cantor base representation* of n, which is a direct extension of the standard base 10 representation of n.

Throughout the paper, n will denote a positive integer. Let $p_1 < p_2 < \cdots < p_\ell$ be the primes dividing n and let $q_1 < q_2 < \cdots$ be the primes not dividing n.

2. THE ALGORITHM

Jones gives the following recursive algorithm for finding a reduced ϕ-partition for n:

1. If n is simple, then $n = 1 * n$ is a reduced ϕ-partition.
2. If $p^2 | n$ for some prime p, then $p * (n/p)$ is a ϕ-partition of n. Apply the algorithm to n/p to give a reduced ϕ-partition $\sum_{j=1}^{i} a_j * s_j$ for n/p; the desired reduced ϕ-partition for n is $\sum_{j=1}^{i} (a_j p) * s_j$.
3. If n is square-free and not simple, then let p be a prime divisor of n and let q be a prime such that $q < p$ and $q \nmid n$. Such p and q exist since n is not simple; p could be chosen to be the largest prime dividing n, and q could be chosen to be the smallest prime not dividing n. Then $(p-q) * (n/p) + 1 * (qn/p)$ is a ϕ-partition for n. Apply the algorithm to n/p and qn/p to give reduced ϕ-partitions $\sum_{j=1}^{i} a_j * s_j$ and $\sum_{j=1}^{i} a'_j * s_j$, respectively. The desired reduced ϕ-partition for n is $\sum_{j=1}^{i} [(p-q)a_j + a'_j] * s_j$.

At each step of the algorithm, it will be generally true that more than one prime or pair of primes can be chosen. The next section shows that the result of the algorithm is independent of these choices.

3. THE ALGORITHM GIVES A UNIQUE REDUCED ϕ-PARTITION

For any integer w, let $\phi_w(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} (1 - w/p)$, so that $\phi_0(n) = n$ and $\phi_1(n) = \phi(n)$. Define a ϕ_w-partition and a reduced ϕ_w-partition analogously to a ϕ-partition and a reduced ϕ-partition, respectively.

If p is prime and $p^2 | n$, then $\phi_w(n) = p\phi_w(n/p)$, and so $p * (n/p)$ is a ϕ_w-partition of n. If p and q are primes, $p|n$, $p^2 \nmid n$, $q \nmid n$ and $p > q$, then

$$\begin{aligned}\phi_w(n) &= (p-w)\phi_w(n/p)\\ &= (p-q)\phi_w(n/p) + (q-w)\phi_w(n/p)\\ &= (p-q)\phi_w(n/p) + \phi_w(qn/p),\end{aligned}$$

so that $(p-q) * (n/p) + 1 * (qn/p)$ is a ϕ_w-partition of n. These facts together with an induction argument show that any reduced ϕ-partition given by the algorithm is also a reduced ϕ_w-partition for every integer w.

For the rest of the paper, let i be the unique positive integer such that $s_i \le n < s_{i+1}$; any reduced ϕ-partition of n can be written in the form $\sum_{j=1}^{i} b_j * s_j$.

Theorem 1: The reduced ϕ-partition for n given by the algorithm above is independent of the primes chosen at each step of the algorithm.

Proof: Suppose that $\sum_{j=1}^{i} a_j * s_j$ is a reduced ϕ-partition of n given by the algorithm. By replacing w with P_1, with P_2, ..., and finally with P_i, the following system of i equations and i unknowns is obtained:

$$\begin{aligned}a_1\phi_{P_1}(s_1) + a_2\phi_{P_1}(s_2) + \cdots + a_i\phi_{P_1}(s_i) &= \phi_{P_1}(n),\\ a_1\phi_{P_2}(s_1) + a_2\phi_{P_2}(s_2) + \cdots + a_i\phi_{P_2}(s_i) &= \phi_{P_2}(n),\\ &\vdots\\ a_1\phi_{P_i}(s_1) + a_2\phi_{P_i}(s_2) + \cdots + a_i\phi_{P_i}(s_i) &= \phi_{P_i}(n).\end{aligned}$$

This system of equations can be rewritten in the form $N\vec{a} = \vec{b}$, where the matrix N has entries

$$N_{\ell k} = \phi_{P_\ell}(s_k) = \prod_{j=1}^{k-1}(P_j - P_\ell).$$

If $\ell < k$, then $N_{\ell k} = 0$, so that N is lower-triangular, and if $\ell = k$, then $N_{\ell k} \ne 0$, so that N is invertible. It follows that the solution to this linear system uniquely determines the coefficients a_j in the reduced ϕ-partition given by the algorithm. □

The algorithm gives a unique reduced ϕ-partition for n, but frequently this is not the only reduced ϕ-partition that n has. The integer 8, for example, has 2 reduced ϕ-partitions: $4 * 2$ and $2 * 1 + 6$. Certain characteristics of the reduced ϕ-partition given by the algorithm are critical, however, in determining whether n has a unique reduced ϕ-partition. The following two theorems summarize these characteristics. Let $M(n) = n/\left(\prod_{\substack{p|n \\ p \text{ prime}}} p\right)$.

Theorem 2: Let k be the largest integer such that $s_k | n$ and let ℓ be the number of distinct prime factors of n. The algorithm above gives a reduced ϕ-partition for n of the form $\sum_{j=k}^{\ell+1} a_j * s_j$, where $a_{\ell+1} = M(n)$ and $a_k \ge a_{k+1} \ge \cdots \ge a_{\ell+1}$.

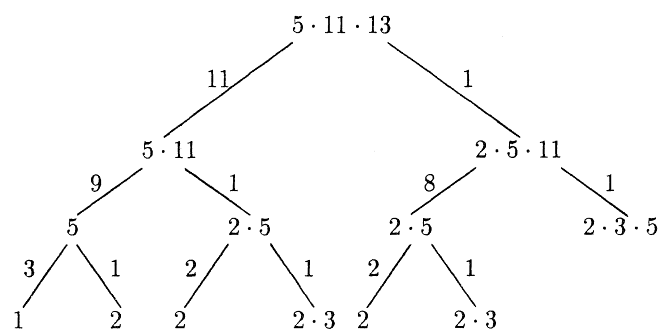
Proof: It follows that $a_j = 0$ for $1 \le j < k$ from examining the first $k-1$ equations of the linear system above and noting that $b_j = \phi_{P_j}(n) = 0$ for $1 \le j < k$. It is clear that $a_{\ell+1} = M(n)$ and

$a_j = 0$ for $\ell+1<j$ from the three cases presented in the algorithm together with an induction argument on n.

The claim that $a_k \geq a_{k+1} \geq \cdots \geq a_{\ell+1}$ will also be proven by induction on n. If $n=1$, then $1=1*1$ is the reduced ϕ-partition. If $n>1$ and $p^2|n$ for some prime p, then establish the claim by using the ϕ-partition $n = p*(n/p)$ and applying the induction hypothesis to n/p.

If n is square-free, then the proof is divided into two cases. In the first case, $p_\ell < q_1$, so that $n = s_k$. If $p_\ell > q_1$ and there is a prime t such that $t \nmid n$ and $q_1 < t < p_\ell$, then the claim follows from using the ϕ-partition $n = (p_\ell - t)*(n/p_\ell) + 1*(tn/p_\ell)$ and applying the induction hypothesis to n/p_ℓ and tn/p_ℓ. If $p_\ell > q_1$, but there is no prime t as described above, then $q_1 n/p_\ell = s_{\ell+1}$, since $q_1 n/p_\ell$ is simple and has the same number of prime factors as n. Prove the claim by using the ϕ-partition $n = (p_\ell - q_1)*(n/p_\ell) + 1*s_{\ell+1}$ and applying the induction hypothesis to n/p_ℓ. □

By Theorem 1, the reduced ϕ-partition $n = \sum_{j=1}^{i} a_j * s_j$ given by the algorithm can be represented by a weighted binary tree as follows. As noted in part 3 of the algorithm, it is possible to choose p as the largest prime dividing n and choose q as the smallest prime dividing n. Assume without loss of generality that n is square-free; the algorithm will find a reduced ϕ-partition for $\prod_{\substack{p|n \\ p \text{ prime}}} p$ and incorporate this reduced ϕ-partition into the ϕ-partition $n = M(n) * \prod_{\substack{p|n \\ p \text{ prime}}} p$. If n is not simple, then the left branch has weight $p_\ell - q_1$ and the left child is n/p_ℓ, while the right branch has weight 1 and the right child is $q_1 n/p_\ell$. Apply this process recursively to n/p_ℓ and $q_1 n/p_\ell$, terminating only when all the leaves of the tree are simple integers. The example below gives the tree for $5\cdot 11\cdot 13$.



It is possible to determine a_j from the tree representation by taking the sum over all paths from n to s_j of the product of the weights along each path. The coefficient a_2 for $5\cdot 11\cdot 13$, for example, is $11\cdot 9\cdot 1 + 11\cdot 1\cdot 2 + 1\cdot 8\cdot 2$.

Let $n = p_1 \ldots p_\ell$, and suppose that m is a vertex at level $u = u(m)$ of the tree described above, where level 0 denotes the top of the tree. Let $L = L(m)$ and $R = R(m) = u = L(m)$ be the number of left and right branches, respectively, in the path from n to m, and define $t_0 < t_1 < \cdots < t_{L-1}$ to be the levels where the path branches to the left. An induction argument on the level u proves the following lemma.

Lemma 3: If m is a vertex at level u of the tree, then $m = p_1 \ldots p_{\ell-u} q_1 \ldots q_R$ and the product of the weights along the path from n to m is $\prod_{e=0}^{L-1}(p_{\ell-t_e} - q_{1+t_e-e})$.

Proof: The proof will be by induction on u. If $u=0$, then $m=n$ and the result is clear. Suppose that the lemma is true for m and its ancestors, and consider the children of m. Assume without loss of generality that m is not simple, since simple vertices have no children. To establish the lemma for the children of m, it suffices to show that the largest prime dividing m is $p_{\ell-u}$ and that the smallest prime not dividing m is q_{R+1}. The largest prime dividing m is clearly $\max(p_{\ell-u}, q_R)$, and the smallest prime not dividing m is clearly $\min(p_{\ell-u+1}, q_{R+1})$. These two facts reduce the two assertions above to the condition $p_{\ell-u} > q_{R+1}$. Divide the proof of this condition into two cases. In the first case, assume that m is the right child of $m' = p_1 \dots p_{\ell-u+1} q_1 \dots q_{R-1}$. It follows from the construction of m that $p_{\ell-u+1}$ is the largest prime dividing m' and that q_R is the smallest prime not dividing m'. It also follows that $p_{\ell-u+1} > q_R$, since m' would be simple otherwise. Hence $p_{\ell-u} > q_{R+1}$, because m is not simple. The proof when m is a left child is similar. □

Fix a path from n to s_j. A left child has one less prime divisor than its parent, while a right child has the same number of prime divisors as its parent. This implies that the path from n to s_j must branch to the left $\ell - j + 1$ times, since s_j has $j-1$ prime factors. Let m be the vertex at level $t_{\ell-j}$. It follows from the proof of the previous lemma that $p_{\ell-t_{\ell-j}}$ is the largest prime dividing m, that $q_{1+t_{\ell-j}-(\ell-j)}$ is the smallest prime not dividing m, and that $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$.

Conversely, suppose that $0 \le t_0 < t_1 < \cdots < t_{\ell-j} < \ell$ and that $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$. An induction argument on $\ell - j$ shows that there is a path from n to s_j that branches to the left at levels $t_0, \dots, t_{\ell-j}$. If $\ell - j = 0$, then an induction on the level together with the previous lemma and the condition $p_{\ell-t_0} > q_{1+t_0}$ guarantees that there is a path branching to the right at levels $0, \dots, t_0 - 1$. This condition also guarantees that the vertex at level t_0 is not simple, since $p_{\ell-t_0}$ and q_{1+t_0} are, respectively, the largest prime dividing the vertex and the smallest prime not dividing the vertex. Branching to the left at level t_0 will give a vertex with $\ell - 1$ prime factors, and so branching to the right at level $t_0 + 1$ and all higher levels will give a path that terminates at s_ℓ.

Assume that the claim above is true for $\ell - j - 1$. This implies that there is a path from n to s_{j+1} that branches to the left at levels $t_0, \dots, t_{\ell-j-1}$, since $p_{\ell-t_{\ell-j-1}} > p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)} \ge q_{1+t_{\ell-j-1}-(\ell-j-1)}$. Take this path from n to level $t_{\ell-j-1}$, and then construct a path from the vertex at this level to s_j by the same method as in the previous paragraph. This proves the claim above. Non-square-free n adds a factor of $M(n)$ to the calculations, as noted previously, and so the claim above combined with the previous lemma proves the following theorem.

Theorem 4: If $k \le j \le \ell$, then

$$a_j = M(n) \Sigma \prod_{e=0}^{\ell-j} (p_{\ell-t_e} - q_{1+t_e-e}),$$

where the sum is taken over all $0 \le t_0 < \cdots < t_{\ell-j} < \ell$ with $p_{\ell-t_{\ell-j}} > q_{1+t_{\ell-j}-(\ell-j)}$. In particular,

$$a_k = M(n) \prod_{e=k}^{\ell} (p_e - q_1) \quad \text{and} \quad a_{k+1} \ge M(n) \prod_{e=k+1}^{\ell} (p_e - q_1).$$

The following section gives a necessary and sufficient criterion for determining if n has a unique reduced ϕ-partition by using the previous two theorems together with the specific Cantor base representation of n described in Section 1.

4. WHEN DOES n HAVE A UNIQUE REDUCED ϕ-PARTITION?

Theorem 5: A positive integer n has a unique reduced ϕ-partition if and only if $n = 9$, n is prime, or the Cantor base representation for n is a reduced ϕ-partition.

By enumerating all possible partitions of 9 consisting of simple integers, one can verify that $3*1+3*2$ is the unique reduced ϕ-partition of 9. The following two lemmas will complete the proof of the *if* part of the theorem.

Lemma 6: Primes have a unique reduced ϕ-partition.

Proof: From Theorem 7 in [1], a ϕ-partition of a prime q must be of the form $(q-r)*1+r$, where r is prime and $r \le q$. The only simple prime number is 2, and so $(q-2)*1+1*2$ is the unique reduced ϕ-partition of q. □

Lemma 7: Let $\sum_{j=1}^{i} c_j * s_j$ be the Cantor base representation of n described in Section 1. If $\sum_{j=1}^{i} c_j * s_j$ is a reduced ϕ-partition, then it is the unique reduced ϕ-partition for n.

Proof: It suffices to show that for any other partition $\sum_{j=1}^{i} b_j * s_j$ of n into simple integers, $\sum_{j=1}^{i} c_j\phi(s_j) < \sum_{j=1}^{i} b_j\phi(s_j)$. Suppose that $\sum_{j=1}^{i} b_j * s_j$ is a counterexample with $\sum_{j=1}^{i} b_j$ minimal. There is an h such that $b_h \ge P_h$ since $\sum_{j=1}^{i} b_j * s_j$ is distinct from $\sum_{j=1}^{i} c_j * s_j$. Form a new partition $\sum_{j=1}^{i} b'_j * s_j$ by converting P_h of the s_h's into an s_{h+1}. This new partition is a counterexample that contradicts the minimality of $\sum_{j=1}^{i} b_j$, and hence proves the lemma. □

Suppose that $n \ne 9$, that n is composite, and that the Cantor base representation for n is not a reduced ϕ-partition. It then follows from Theorem 2 that $a_k \ge P_k$, where $\sum_{j=1}^{i} a_j * s_j$ is the reduced ϕ-partition given by the algorithm and k the largest positive integer such that $s_k | n$, as defined in Theorem 2. Theorem 4 gives the formula $a_k = M(n)\prod_{e=k}^{\ell}(p_e - P_k)$, since $q_1 = P_k$. It is clear from this formula that $a_k \ne P_k$, since P_k is prime, P_k does not divide n, and P_k does not divide $(p - P_k)$ for any primes p dividing n. If $k > 1$, then apply the following lemma with $h = k$ to show that n has a second reduced ϕ-partition. If $k = 1$, then n is odd, $q_1 = 2$, and the inequality $a_2 \ge M(n)\prod_{e=2}^{\ell}(p_e - 2)$ is a result of Theorem 4. It is a straightforward consequence of this inequality that $a_2 > 3$ if $n \ne 15$, and so it follows from the following lemma with $h = 2$ that n has a second reduced ϕ-partition. The observation that $3+1+3*2+6$ and $1+7*2$ are reduced ϕ-partitions for 15 completes the proof of the theorem.

Lemma 8: Let $\sum_{j=1}^{i} b_j * s_j$ be a reduced ϕ-partition of n. If $b_h > P_h$ for some $h > 1$, then n has a second reduced ϕ-partition.

Proof: To prove the lemma, it will be necessary to show that for each $j > 1$ there is a partition $\sum_{f=1}^{j-1} \beta_f * s_f$ of s_j such that $\sum_{f=1}^{j-1} \beta_f\phi(s_f) = 2\phi(s_j)$. This will be shown by induction on j. If $j = 2$, then $2 = 2*1$ is the desired partition. If $j > 2$, then, by the induction hypothesis, s_{j-1} has a partition $\sum_{f=1}^{j-2} \beta'_f * s_f$ with the desired property. Hence, $s_j = \sum_{f=1}^{j-2}[\beta'_f(P_{j-1}-2)] * s_f + 2 * s_{j-1}$ is a partition of s_j with the desired property.

Now, suppose h is a positive integer such as in the hypothesis of the lemma and let $\sum_{f=1}^{h-1}\beta_f * s_f$ be a partition of s_h with the above property. Construct a new reduced ϕ-partition for n by combining P_h of the s_h terms into one s_{h+1} term. There is a net loss of $\phi(s_h)$ when the sum of the ϕ values in the partition is taken, since $\phi(s_{h+1}) = (P_h - 1)\phi(s_h)$. Breaking up one of the remaining s_h terms compensates for this loss. The second reduced ϕ-partition for n is

$$\sum_{f=1}^{h-1}(b_f + \beta_f) * s_f + (b_h - P_h - 1) * s_h + (b_{h+1} + 1) * s_{h+1} + \sum_{f=h+2}^{i} b_f * s_f. \quad \square$$

REFERENCE

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