THE ASYMPTOTIC BEHAVIOR OF THE GOLDEN NUMBERS

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In [2] the "Golden polynomials"

$$G_{n+2}(x) = xG_{n+1}(x) + G_n(x), \quad G_0(x) = -1, \; G_1(x) = x - 1,$$

and their maximal real root g_n (the "golden numbers") were investigated. It was observed that, as $n \to \infty$, $g_n \to 3/2$; furthermore, it was suggested there might be a more precise formula, since numerical experiments seemed to indicate a dependency on the parity of n of the lower order terms.

This open question will be solved in the present paper.

Solving the recursion for the Golden polynomials by standard methods, we get the explicit formula

$$G_n(x) = A\lambda^n + B\mu^n,$$

with

$$\lambda = \frac{x + \sqrt{x^2+4}}{2}, \quad \mu = \frac{x - \sqrt{x^2+4}}{2},$$
$$A = \frac{1}{2\sqrt{x^2+4}}\left(3x - 2 - \sqrt{x^2+4}\right), \quad B = -\frac{1}{2\sqrt{x^2+4}}\left(3x - 2 + \sqrt{x^2+4}\right).$$

Everything is much nicer when we substitute

$$x = u - \frac{1}{u}.$$

$G_n(x) = 0$ can be rephrased as $-B/A = (\lambda/\mu)^n$, or

$$\frac{(2u+1)(u-1)}{(u+1)(u-2)} = (-u^2)^n.$$

Now it is plain to see that, for large n, this equation can only hold if u is either close to 2 or to $u = -1/2$. In both cases, this would mean x is close to 3/2. Let us assume that u is close to 2. It is clear that the cases when n is even or odd have to be distinguished. We start with $n = 2m$ and rewrite the equation as

$$u - 2 = \frac{(2u+1)(u-1)}{(u+1)} u^{-4m}.$$

We get the asymptotic behavior of the desired solution by a process known as "bootstrapping" which is explained in [1]. First, we set $u = 2 + \delta$, insert $u = 2$ into the right-hand side, and get an approximation for δ. Then we insert $u = 2 + \delta$ into the right-hand side, expand, and get the next term. This procedure can be repeated to get as many terms as needed. In this way, we get

$$\delta \sim \frac{5}{3} \cdot 16^{-m},$$

and with

$$u = 2 + \frac{5}{3} \cdot 16^{-m} + \varepsilon,$$

we find

$$\varepsilon \sim -\frac{25}{6} m \cdot 256^{-m}.$$

From

$$u \sim 2 + \frac{5}{3} \cdot 16^{-m} - \frac{50}{9} m \cdot 256^{-m}$$

we find by substitution

$$x \sim \frac{3}{2} + \frac{25}{12} \cdot 16^{-m} - \frac{125}{18} m \cdot 256^{-m}.$$

Now let us consider the case n is odd, $n = 2m+1$. Then our equation is

$$u - 2 = -\frac{(2u+1)(u-1)}{(u+1)u^2} u^{-4m},$$

and we find as above

$$u \sim 2 - \frac{5}{12} \cdot 16^{-m} - \frac{25}{72} m \cdot 256^{-m},$$

and also

$$x \sim \frac{3}{2} - \frac{25}{48} \cdot 16^{-m} - \frac{125}{288} m \cdot 256^{-m}.$$

Confining ourselves to two terms, we write our findings in a single formula as

$$g_n \sim \frac{3}{2} + (-1)^n \frac{25}{12} \cdot 4^{-n},$$

which matches perfectly with the empirical data from [2].

REFERENCES

1. D. Greene & D. Knuth. *Mathematics for the Analysis of Algorithms.* Birkhäuser, 1981.
2. G. Moore. "The Limit of the Golden Numbers is 3/2." *The Fibonacci Quarterly* **32.3** (1994): 211-17.

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