ON A BINOMIAL SUM FOR THE FIBONACCI AND RELATED NUMBERS

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1. INTRODUCTION

Let u and v be nonzero integers, and let r be an integer. It is well known that

$$F_{un+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{vk+r}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

if and only if

$$s = F_u / F_v, \quad t = (-1)^u F_{v-u} / F_v. \tag{2}$$

This result originates with Carlitz [2], and was recently interpreted via exponential generating functions (or egf's) by Prodinger [7]. The purpose of this paper is to show that the egf method is also an efficient tool in deriving similar results for the Lucas numbers L_n, the Pell numbers P_n, and the Pell-Lucas numbers R_n.

The egf of a sequence $\{a_n\}$ is defined by

$$\hat{a}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

The product of the egf's of $\{a_n\}$ and $\{b_n\}$ generates the binomial convolution of $\{a_n\}$ and $\{b_n\}$:

$$\hat{a}(x)\hat{b}(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_{n-k} b_k \right) \frac{x^n}{n!}. \tag{3}$$

The right side of (1) is thus the binomial convolution of the sequences $\{t^n\}$ and $\{s^n F_{vn+r}\}$. The egf of the geometric progression $\{t^n\}$ is e^{tx}.

The proofs of this note are based on the following two lemmas.

Lemma 1: Let λ_1 and λ_2 be given distinct complex numbers, and let c_1 and c_2 be given nonzero *distinct* complex numbers. Then

$$c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}$$

if and only if

$$\mu_1 = \lambda_1 \quad \text{and} \quad \mu_2 = \lambda_2.$$

Lemma 2: Let λ_1 and λ_2 be given distinct complex numbers, and let c be a given nonzero complex number. Then

$$c e^{\lambda_1 x} + c e^{\lambda_2 x} = c e^{\mu_1 x} + c e^{\mu_2 x}$$

if and only if either

$$\mu_1 = \lambda_1 \quad \text{and} \quad \mu_2 = \lambda_2$$

or

$$\mu_1 = \lambda_2 \quad \text{and} \quad \mu_2 = \lambda_1.$$

The lemmas follow from the linear independence of the functions $e^{\lambda x}$.

Lemma 1 is needed for the Fibonacci and the Pell numbers, and for the Lucas and the Pell-Lucas numbers in the case $r \neq 0$, while Lemma 2 is needed for the Lucas and the Pell-Lucas numbers in the case $r = 0$. We do not consider Fibonacci numbers here, since the egf method is applied to them in [7].

For a general account on egf's we refer to [4], and for egf's of Fibonacci and Lucas sequences we refer to [3], [5], and [6].

2. ON THE LUCAS NUMBERS

Let the negative index Fibonacci and Lucas numbers be defined by $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ $(n \geq 0)$. Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. The well-known Binet form of the Lucas numbers is $L_n = \alpha^n + \beta^n$. Thus, it is easy to see that

$$\hat{L}(x) = e^{\alpha x} + e^{\beta x}.$$

We now state the promised binomial results for the Lucas numbers. We distinguish two cases: $r \neq 0$ and $r = 0$.

Theorem 1: Let u and v be nonzero integers, and let r be a *nonzero* integer. Then

$$L_{un+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k L_{vk+r}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

if and only if

$$s = F_u / F_v, \quad t = (-1)^u F_{v-u} / F_v. \tag{5}$$

Proof: In terms of the egf's, (4) can be written as

$$\alpha^r e^{\alpha^u x} + \beta^r e^{\beta^u x} = e^{tx}(\alpha^r e^{\alpha^v sx} + \beta^r e^{\beta^v sx}), \tag{6}$$

where the right side comes from the property (3). Since $r \neq 0$, we have $\alpha^r \neq \beta^r$. Thus, by Lemma 1, (6) holds if and only if

$$\alpha^u = t + \alpha^v s, \quad \beta^u = t + \beta^v s, \tag{7}$$

that is,

$$s = \frac{\alpha^u - \beta^u}{\alpha^v - \beta^v} = \frac{F_u}{F_v}, \quad \text{and} \quad t = \alpha^u - \alpha^v \frac{\alpha^u - \beta^u}{\alpha^v - \beta^v} = (-1)^u \frac{F_{v-u}}{F_v},$$

where the last equality follows from the property $\alpha\beta = -1$. This completes the proof of Theorem 1.

Remark: Note that (5) is equivalent to (2).

Theorem 2: Let u and v be nonzero integers (and $r=0$). Then

$$L_{un}=\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^{k}L_{vk}, \quad n=0,1,2,\ldots, \tag{8}$$

if and only if either (5) holds or

$$s=-F_u/F_v, \quad t=F_{u+v}/F_v. \tag{9}$$

Proof: In terms of the egf's, (8) can be written as

$$e^{\alpha^u x}+e^{\beta^u x}=e^{tx}(e^{\alpha^v sx}+e^{\beta^v sx}), \tag{10}$$

where the right side comes from property (3). By *Lemma 2*, (10) holds if and only if either (7) holds or

$$\alpha^u=t+\beta^v s, \quad \beta^u=t+\alpha^v s. \tag{11}$$

By the proof of Theorem 1, (7) is equivalent to (5). On the other hand, (11) holds if and only if

$$s=\frac{\beta^u-\alpha^u}{\alpha^v-\beta^v}=\frac{-F_u}{F_v}, \quad \text{and} \quad t=\alpha^u-\beta^v\frac{\beta^u-\alpha^u}{\alpha^v-\beta^v}=\frac{F_{u+v}}{F_v}.$$

This completes the proof of Theorem 2.

Corollary 1: If u and v are nonzero integers and r is an integer, then

$$F_v^n L_{un+r}=\sum_{k=0}^{n}\binom{n}{k}(-1)^{u(n-k)}F_{v-u}^{n-k}F_u^k L_{vk+r},$$

$$F_v^n L_{un}=\sum_{k=0}^{n}\binom{n}{k}F_{u+v}^{n-k}(-1)^k F_u^k L_{vk}.$$

Corollary 2: If u is a nonzero integer and r is an integer, then

$$L_{un+r}=\sum_{k=0}^{n}\binom{n}{k}F_{u-1}^{n-k}F_u^k L_{k+r},$$

$$L_{un}=\sum_{k=0}^{n}\binom{n}{k}F_{u+1}^{n-k}(-1)^k F_u^k L_k.$$

Corollary 3: If r is an integer, then

$$L_{2n+r}=\sum_{k=0}^{n}\binom{n}{k}L_{k+r},$$

$$L_{2n}=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^k L_k.$$

Corollary 1 follows from Theorems 1 and 2. Corollary 2 is Corollary 1 with $v=1$, and Corollary 3 is Corollary 2 with $u=2$. Note that the first identities in Corollaries 1-3 also hold for $r=0$, cf. equation (5) in Theorem 2.

3. ON THE PELL NUMBERS

The Pell numbers P_n are defined by

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}, \quad n = 2, 3, \ldots,$$
$$P_{-n} = (-1)^{n+1} P_n, \quad n = 1, 2, \ldots.$$

The well-known Binet form of the Pell numbers is

$$P_n = \frac{a^n - b^n}{a - b},$$

where $a = 1 + \sqrt{2}$, $b = 1 - \sqrt{2}$, that is, where a and b are the roots of the equation $y^2 = 2y + 1$, see, e.g., [1]. Note that $a + b = 2$, $ab = -1$, and $a - b = 2\sqrt{2}$. Using the Binet form, it is easy to see that

$$\hat{P}(x) = \frac{1}{2\sqrt{2}}(e^{ax} - e^{bx}).$$

The Pell numbers have many properties similar to those of the Fibonacci numbers. We here point out that a property analogous to that given in (1) and (2) holds for the Pell numbers. As in the case of the Fibonacci numbers, we need not distinguish the cases $r \neq 0$ and $r = 0$ here.

Theorem 3: Let u and v be nonzero integers, and let r be an integer. Then

$$P_{un+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k P_{vk+r}, \quad n = 0, 1, 2, \ldots, \tag{12}$$

if and only if

$$s = P_u / P_v, \quad t = (-1)^u P_{v-u} / P_v. \tag{13}$$

Proof: In terms of the egf's, (12) is

$$\frac{1}{2\sqrt{2}}(a^r e^{a^u x} - b^r e^{b^u x}) = e^{tx} \frac{1}{2\sqrt{2}}(a^r e^{a^v sx} - b^r e^{b^v sx}). \tag{14}$$

Since $a^r \neq -b^r$ for all r, we may apply *Lemma 1*. Thus (14) holds if and only if

$$a^u = t + a^v s, \quad b^u = t + b^v s, \tag{15}$$

which can be shown to hold if and only if (13) holds; cf. the proof of (5). The last equality in (13) follows from the property $ab = -1$. This completes the proof of Theorem 3.

Corollary 4: If u and v are nonzero integers and r is an integer, then

$$P_v^n P_{un+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{u(n-k)} P_{v-u}^{n-k} P_u^k P_{vk+r}.$$

Corollary 5: If u is a nonzero integer and r is an integer, then

$$P_{un+r} = \sum_{k=0}^{n} \binom{n}{k} P_{u-1}^{n-k} P_u^k P_{k+r}.$$

Corollary 6: If r is an integer, then

$$P_{2n+r} = \sum_{k=0}^{n} \binom{n}{k} 2^k P_{k+r}.$$

4. ON THE PELL-LUCAS NUMBERS

The numbers R_n are defined by

$$R_0 = 2,\ R_1 = 2,\ R_n = 2R_{n-1} + R_{n-2}, \quad n = 2, 3, \ldots,$$
$$R_{-n} = (-1)^n R_n, \quad n = 1, 2, \ldots.$$

These numbers are associated with the Pell numbers in a way similar to that in which the Lucas numbers are associated with the Fibonacci numbers, see, e.g., [1]. Therefore, we refer to the numbers R_n as the Pell-Lucas numbers. The Pell-Lucas numbers have the Binet form $R_n = a^n + b^n$, where a and b are as in Section 3. Thus,

$$\hat{R}(x) = e^{ax} + e^{bx}.$$

The Pell-Lucas numbers possess the properties of the Lucas numbers given in Theorems 1 and 2. We state these properties in Theorems 4 and 5. The proofs of Theorems 4 and 5 are similar to those of Theorems 1 and 2, and are omitted for brevity.

Theorem 4: Let u and v be nonzero integers, and let r be *nonzero* integer. Then

$$R_{un+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k R_{vk+r}, \quad n = 0, 1, 2, \ldots, \tag{16}$$

if and only if

$$s = P_u / P_v, \quad t = (-1)^u P_{v-u} / P_v. \tag{17}$$

Remark: Note that (17) is equivalent to (13).

Theorem 5: Let u and v be nonzero integers (and $r = 0$). Then

$$R_{un} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k R_{vk}, \quad n = 0, 1, 2, \ldots, \tag{18}$$

if and only if either (17) holds or

$$s = -P_u / P_v, \quad t = P_{u+v} / P_v. \tag{19}$$

5. REMARK

It may be worth recalling that the egf method is, of course, also a very efficient tool in deriving other binomial identities. We mention here two such identities, namely,

$$\sum_{k=0}^{n} \binom{n}{k} F_{uk+r} L_{u(n-k)+r} = 2^n F_{un+2r}, \tag{20}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}P_{uk+r}R_{u(n-k)+r}=2^n P_{un+2r}. \tag{21}$$

The left side of (20) can be written in terms of egf's as

$$\frac{1}{\sqrt{5}}(\alpha^r e^{\alpha^u x}-\beta^r e^{\beta^u x})(\alpha^r e^{\alpha^u x}+\beta^r e^{\beta^u x})$$

and the right side as

$$\frac{1}{\sqrt{5}}(\alpha^{2r} e^{\alpha^u 2x}-\beta^{2r} e^{\beta^u 2x}).$$

It is clear that these two egf's are equal; hence, (20) holds. The proof of (21) is similar.

For further examples, reference is made to [3], [5], and [6].

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