POLYNOMIALS ASSOCIATED WITH GENERALIZED MORGAN-VOYCE POLYNOMIALS

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1. PROLOGUE

André-Jeannin [1] recently defined a polynomial sequence $\{P_n^{(r)}(x)\}$, where r is a real number, by the recurrence

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{1.1}$$

with

$$P_0^{(r)}(x) = 1, \quad P_1^{(r)}(x) = x + r + 1. \tag{1.2}$$

Furthermore [1], a sequence of integers $\{a_{n,k}^{(r)}\}$ exists for which

$$P_n^{(r)}(x) = \sum_{k=0}^{n} a_{n,k}^{(r)} x^k, \tag{1.3}$$

where

$$a_{n,n}^{(r)} = 1 \quad (n \geq 0). \tag{1.4}$$

He also proved [1] the crucial formula ($n \geq 0$, $k \geq 0$)

$$a_{n,k}^{(r)} = \binom{n+k}{2k} + r\binom{n+k}{2k+1} \tag{1.5}$$

and the recurrence

$$a_{n,k}^{(r)} = 2a_{n-1,k}^{(r)} - a_{n-2,k}^{(r)} + a_{n-1,k-1}^{(r)} \quad (n \geq 2, k \geq 1). \tag{1.6}$$

Simple instances of $P_n^{(r)}(x)$ are [1], with slightly varied notation,

$$P_{n-1}^{(0)}(x) = b_n(x) \quad (n \geq 1) \tag{1.7}$$

and

$$P_{n-1}^{(1)}(x) = B_n(x) \quad (n \geq 1), \tag{1.8}$$

where $b_n(x)$ and $B_n(x)$ are the well-known *Morgan-Voyce polynomials* [4]. (Please see [1] for other references to $b_n(x)$ and $B_n(x)$.)

It is the purpose of this short paper to give a brief account of a closely related sequence of polynomials $\{Q_n^{(r)}(x)\}$ with particular emphasis on the case $r = 0$. Necessarily, a formula corresponding to (1.5) will have to be discovered.

For ready comparison and contrast with the contents of [1], it seems desirable to present this material in a partially similar way. Before proceeding, however, we need to add the following items of information.

Lemma 1: $a_{n,k}^{(1)} - a_{n-2,k}^{(1)} = a_{n,k}^{(0)} + a_{n-1,k}^{(0)} \quad (n \geq 2)$.

Proof: $\binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+k+1}{2k+1}$ by Pascal's Theorem,

i.e., $\binom{n+k}{2k} + \binom{n+k-1}{2k} + \binom{n+k-1}{2k+1} = \binom{n+k+1}{2k+1}$ by Pascal's Theorem,

i.e., $\binom{n+k}{2k} + \binom{n+k-1}{2k} = \binom{n+k+1}{2k+1} - \binom{n+k-1}{2k+1}$.

Use (1.5) for $r = 0, r = 1$, and the Lemma follows by Pascal's Theorem.

When $r = 2$ in (1.1), then $P_{n-1}^{(2)}(x)$ is found to be

$$P_{n-1}^{(2)}(x) = c_n(x) = \frac{b_{n+1}(x) - b_{n-1}(x)}{x} \quad (n \geq 1), \tag{1.9}$$

where $c_n(x)$—given in terms of Morgan-Voyce polynomials—has been introduced independently by me is a paper currently being written in which it is also demonstrated that

$$c_{n+1}(x) - c_n(x) = C_n(x), \tag{1.10}$$

in which $C_n(x)$ is to be defined in (2.11).

2. THE POLYNOMIALS $\{Q_n^{(r)}(x)\}$

Define, as in (1.1), a polynomial sequence $\{Q_n^{(r)}(x)\}$ recursively by

$$Q_n^{(r)}(x) = (x+2)Q_{n-1}^{(r)}(x) - Q_{n-2}^{(r)}(x) \quad (n \geq 2) \tag{2.1}$$

with

$$Q_0^{(r)}(x) = 2, \quad Q_1^{(r)}(x) = x + r + 2. \tag{2.2}$$

Then a sequence of integers $\{b_{n,k}^{(r)}\}$ exists such that

$$Q_n^{(r)}(x) = \sum_{k=0}^{n} b_{n,k}^{(r)} x^k, \tag{2.3}$$

where

$$b_{n,n}^{(r)} = \begin{cases} 1 & (n \geq 1), \\ 2 & (n = 0). \end{cases} \tag{2.4}$$

Now $b_{n,0}^{(r)} = Q_n^{(r)}(0)$. By (2.1) and (2.2),

$$b_{n,0}^{(r)} = 2b_{n-1,0}^{(r)} - b_{n-2,0}^{(r)} \quad (n \geq 2) \tag{2.5}$$

with

$$\begin{cases} b_{0,0}^{(r)} = 2, \\ b_{1,0}^{(r)} = 2 + r \quad \text{by (2.2)}. \end{cases} \tag{2.6}$$

Following [1], we deduce that ($n \geq 0$)

$$b_{n,0}^{(r)} = 2 + nr, \tag{2.7}$$

whence

$$b_{n,0}^{(0)} = 2 \tag{2.8}$$

and

$$b_{n,0}^{(1)} = 2 + n. \tag{2.9}$$

Comparison of coefficients of x^k in (1.1) leads to the recurrence ($n \geq 2$, $k \geq 1$)

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} + b_{n-1,k-1}^{(r)} - b_{n-2,k}^{(r)}. \tag{2.10}$$

Table 1 displays a triangular arrangement of the coefficients $b_{n,k}^{(r)}$. This ought to be compared with the (preferably extended) table in [1] for the coefficients $a_{n,k}^{(r)}$.

TABLE 1. Coefficients $b_{n,k}^{(r)}$ of $Q_n^{(r)}(x)$

$n \backslash k$	0	1	2	3	4	5	...
0	2						...
1	$2+r$	1					...
2	$2+2r$	$4+r$	1				...
3	$2+3r$	$9+4r$	$6+r$	1			...
4	$2+4r$	$16+10r$	$20+6r$	$8+r$	1		...
5	$2+5r$	$25+20r$	$50+21r$	$35+8r$	$10+r$	1	...
...	...	...	...	...	...	...	...

Next, we introduce the important symbolism

$$Q_n^{(0)}(x) = C_n(x). \tag{2.11}$$

Using Table 1, we may now write out the expressions for $C_0(x), C_1(x), C_2(x), C_3(x), \ldots$. Some properties of $C_n(x)$, especially in relation to Lucas polynomials, appear in [2].

3. CONNECTION BETWEEN $\{P_n^{(r)}(x)\}$ AND $\{Q_n^{(r)}(x)\}$

Inherent in the nature of the laws of formation of $\{P_n^{(r)}(x)\}$ and $\{Q_n^{(r)}(x)\}$—namely, (1.1), (1.2), (2.1), and (2.2)—is the inevitably close connection between $a_{n,k}^{(r)}$ and $b_{n,k}^{(r)}$.

Typically, for example,

$$\begin{cases} b_{5,2}^{(r)} = 50 + 21r &= (35 + 21r) + 15 = a_{5,2}^{(r)} + a_{4,2}^{(0)} \\ b_{6,3}^{(r)} = 112 + 36r &= (84 + 36r) + 28 = a_{6,3}^{(r)} + a_{5,3}^{(0)}. \end{cases} \tag{3.1}$$

These illustrations suggest the nature of the constant (for which $r = 0$) by which $b_{n,k}^{(r)}$ exceeds $a_{n,k}^{(r)}$. It is $a_{n-1,k}^{(0)}$.

Theorem 1: $b_{n,k}^{(r)} = a_{n,k}^{(r)} + a_{n-1,k}^{(0)} \quad (n \geq 1)$

$$= \binom{n+k}{2k} + \binom{n-1+k}{2k} + r\binom{n+k}{2k+1} \text{ by (1.5).}$$

Proof: Follow the inductive proof in [1] for $a_{n,k}^{(r)}$, using the binomial coefficients and (2.10). The occurrence of the middle (extra) binomial term causes no complication.

[Alternatively: Subtract (1.6) from (2.10) and use induction.]

Combine the first two binomial coefficients in Theorem 1 to derive

Corollary 1: $b_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}$.

Multiply both sides of Theorem 1 by x^k and sum. Immediately, from (1.3) and (2.3), we infer the fundamental polynomial property associating $Q_n^{(r)}(x)$ with $P_n^{(r)}(x)$.

Theorem 2: $Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x) \quad (n \geq 1)$.

Fixing $r = 0$ in Theorem 2 and using (1.7) and (2.11), we deduce

$$C_n(x) = b_{n+1}(x) + b_n(x). \tag{3.2}$$

Evaluating in Theorem 2 when $x = 1$ produces a nice specialization. Already [1] we know that, for Fibonacci numbers,

$$P_n^{(r)}(1) = F_{2n+1} + rF_{2n}. \tag{3.3}$$

Application of (3.3) enables us to get the following two useful subsidiary results for Fibonacci and Lucas numbers from Theorem 2 when $x = 1$.

Corollary 2: $Q_n^{(r)}(1) = L_{2n} + rF_{2n}$.

Proof:
$$\begin{aligned} Q_n^{(r)}(1) &= P_n^{(r)}(1) + P_{n-1}^{(0)}(1) && \text{by Theorem 2} \\ &= F_{2n+1} + rF_{2n} + F_{2n-1} && \text{by (3.3)} \\ &= L_{2n} + rF_{2n}. \end{aligned}$$

Corollary 3: $Q_n^{(2u+1)}(1) = 2P_n^{(u)}(1)$.

Proof:
$$\begin{aligned} Q_n^{(2u+1)}(1) &= F_{2n+1} + (2u+1)F_{2n} + F_{2n-1} \quad \text{as in Corollary 2 } (r = 2u+1, \text{ odd}) \\ &= 2(F_{2n+1} + uF_{2n}) \\ &= 2P_n^{(u)}(1) \quad \text{by (3.3).} \end{aligned}$$

Thus,

$$Q_n^{(1)}(1) = 2P_n^{(0)}(1) = 2F_{2n+1} = 2b_{n+1}, \tag{3.4}$$

$$Q_n^{(3)}(1) = 2P_n^{(1)}(1) = 2F_{2n+2} = 2B_{n+1}, \tag{3.5}$$

$$Q_n^{(5)}(1) = 2P_n^{(2)}(1) = 2L_{2n+1} = 2c_{n+1}. \tag{3.6}$$

Conventional symbolism $b_n(1) = b_n, \ldots$ has been employed in (3.4)-(3.6). Even superscript values of r in Corollary 2 do not, in general, appear to produce neat or interesting simplifications. However, by Corollary 2, (2.11), and [2], we do know that

$$Q_n^{(0)}(1) = C_n = L_{2n}. \tag{3.7}$$

Worth recording in passing is

$$Q_n^{(2)}(1) = F_{2n+3} = b_{n+2}. \tag{3.8}$$

4. CONNECTION BETWEEN $Q_n^{(0)}(x)$ AND $B_n(x)$

Lastly, the link between our polynomials and the Morgan-Voyce polynomial $B_n(x)$ is described.

Theorem 3: $Q_n^{(0)}(x) = B_{n+1}(x) - B_{n-1}(x)$.

Proof:
$$\begin{aligned}
Q_n^{(0)}(x) &= \sum_{k=0}^{n} b_{n,k}^{(0)} x^k && \text{by (2.3) } (r = 0) && \text{(i)}\\
&= \sum_{k=0}^{n} (a_{n,k}^{(0)} + a_{n-1,k}^{(0)} && \text{by Theorem 1 } (r = 0)\\
&= \sum_{k=0}^{n} (a_{n,k}^{(1)} - a_{n-2,k}^{(1)}) && \text{by Lemma 1}\\
&= P_n^{(1)}(x) - P_{n-2}^{(1)}(x) && \text{by (1.3)}\\
&= B_{n+1}(x) - B_{n-1}(x) && \text{by (1.8).}
\end{aligned}$$

Corollary 4:
$$\begin{aligned}
C_n(x) &= B_{n+1}(x) - B_{n-1}(x) && \text{by (2.11), Theorem 3}\\
&= \sum_{k=0}^{n-1} \frac{n}{k}\binom{n-1+k}{2k-1} x^k + 2 + x^n && \text{by (i), (2.4), (2.8), (2.11), Corollary 1.}
\end{aligned}$$

The property embodied in Corollary 4 means that $B_n(x)$ and $C_n(x)$ form another pair of cognate polynomials which can be incorporated into the synthesis [3], to which all the theory therein applies, e.g.,

$$B_n(x)C_n(x) = B_{2n}(x), \tag{4.1}$$

$$\frac{d}{dx}C_n(x) = nB_n(x). \tag{4.2}$$

5. CHEBYSHEV POLYNOMIALS

Polynomials $P_n^{(r)}(x)$ are shown [1] to be related to $U_n(x)$, the *Chebyshev polynomials of the second kind*. In particular, with an adjusted subscript notation,

$$B_n(x) = \frac{\sin nt}{\sin t} = U_n\left(\frac{x+2}{2}\right), \tag{5.1}$$

where

$$x+2=2\cos t. \tag{5.2}$$

Now, by Theorem 3, Corollary 4, and (5.2),

$$\begin{aligned}
Q_n^{(0)}(x) &= C_n(x) = B_{n+1}(x) - B_{n-1}(x) \\
&= \frac{\sin(n+1)t - \sin(n-1)t}{\sin t} \\
&= 2\cos nt && (5.3) \\
&= 2T_n\left(\frac{x+2}{2}\right), && (5.4)
\end{aligned}$$

where $T_n(x)$ are the *Chebyshev polynomials of the first kind.*

More generally, we construct the law relating $Q_n^{(r)}(x)$ to the two types of Chebyshev polynomials. Needed for this is a pair of known results involving Chebyshev polynomials (our notation):

$$P_n^{(r)}(x) = U_{n+1}\left(\frac{x+2}{2}\right) + (r-1)U_n\left(\frac{x+2}{2}\right) \quad \text{by [1];} \tag{5.5}$$

$$2T_n(x) = U_{n+1}(x) - U_{n-1}(x). \tag{5.6}$$

Theorem 4: $Q_n^{(r)}(x) = 2T_n\left(\frac{x+2}{2}\right) + rU_n\left(\frac{x+2}{2}\right).$

Proof: $Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x)$ by Theorem 2 $(n \geq 1)$

$$\begin{aligned}
&= U_{n+1}\left(\frac{x+2}{2}\right) + (r-1)U_n\left(\frac{x+2}{2}\right) + U_n\left(\frac{x+2}{2}\right) - U_{n-1}\left(\frac{x+2}{2}\right) \quad \text{by (5.5)} \\
&= U_{n+1}\left(\frac{x+2}{2}\right) - U_{n-1}\left(\frac{x+2}{2}\right) + rU_n\left(\frac{x+2}{2}\right) \\
&= 2T_n\left(\frac{x+2}{2}\right) + rU_n\left(\frac{x+2}{2}\right) \quad \text{by (5.6).}
\end{aligned}$$

Zeros

Zeros x_k $(k = 1, 2, \ldots, n)$ of $C_n(x) = Q_n^{(0)}(x)$ are, by (5.4), tied to the zeros of $T_n(\frac{x+2}{2})$. Thus,

$$x_k + 2 = 2\cos\left(\frac{2k-1}{n}\cdot\frac{\pi}{2}\right) \quad (k = 1, 2, \ldots, n)$$

implying

$$x_k = -4\sin^2\left(\frac{2k-1}{2n}\cdot\frac{\pi}{2}\right) \quad (k = 1, 2, \ldots, n). \tag{5.7}$$

For instance, the 3 zeros of $C_3(x)[=2T_3(\frac{x+2}{2})] = x^3 + 6x^2 + 9x + 2 = (x+2)(x^2 + 4x + 1)$ are

$$x_k = -4\sin^2\left(\frac{\pi}{12}\right),\quad -4\sin^2\left(\frac{\pi}{4}\right) = -2,\quad -4\sin^2\left(\frac{5\pi}{12}\right)\quad (k = 1, 2, 3).$$

Zeros of $P_n^{(r)}(x)$ $(r = 0, 1, 2, \ldots, n)$ are given in [1].

EPILOGUE

Together with the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$, the polynomials $c_n(x)$ and $C_n(x)$ constitute an appealing quartet of polynomial relationships which form the subject of my paper alluded to following (1.9). Here, they exhibit a nice simplicity amid complexity, a cohesion and unity amid diversity.

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