ON SEQUENCES RELATED TO EXPANSIONS OF REAL NUMBERS

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1. INTRODUCTION

We intend to study some sequences of real numbers which are obtained as follows: take a natural number N and a real number α and form the sequence $s(N, \alpha) = (a_0, a_1, a_2, \ldots, a_k, \ldots)$, where the numbers a_i are defined by

$$a_0 = \alpha,$$
$$a_n = \begin{cases} 2a_{n-1} & \text{if } 2a_{n-1} < N+n, \\ 2a_{n-1} - (N+n) & \text{otherwise.} \end{cases} \tag{1}$$

The sequences arise from certain nonstandard expansions of real numbers that are discussed in Section 3.

It is very easy to study these sequences *by computer*. This is what we did, and which led us to the following

Conjecture 1: When α is an integer $\in [0, N+2)$, the sequence $s(N, \alpha)$ will *end* in a sequence of zeros.

We verified the truth of this statement for all $N \le 2\,000\,000$.

In the next section we shall show that there is also some "probabilistic" evidence for this conjecture. In Section 3 we shall see that the conjecture has some "heuristic evidence." Finally, we shall conclude with a discussion of some other aspects of the problem.

2. PROBABILISTIC EVIDENCE

Consider a sequence $s(N, \alpha) = (a_0, a_1, a_2, \ldots, a_k, \ldots)$, where N is a natural number ≥ 2 and $\alpha = a_o \in (0, N+2]$ and where the a_k are obtained by the relations (1).

If $\alpha < 0$, then $a_k = 2^k \alpha$; if $\alpha = N+2+\beta$ $(\beta \ge 0)$, then $a_k = N+k+2+2^k$ for all k. Thus, the behavior of $s(N, \alpha)$ is "sufficiently known" for such α.

If $0 \le a < N+2$, then it is easy to show that every a_k is in $[0, N+k+2)$.

Indeed, this is obvious when $k = 0$. Suppose it is true for some $k \ge 0$. Then

- when $a_{k+1} = 2a_k$, we have $a_{k+1} < N+k+1 < N+k+3$,
- when $a_{k+1} = 2a_k - N - k - 1$, then $a_{k+1} \le 2(N+k+2) - N - k - 1 = N+k+3$.

Therefore, our assumption follows by induction.

Now, let α be an *integer* in $(0, N+2)$. Then it is easy to verify that a_k will be in the interval $[0, N+k)$ as soon as $k \ge 2$. Further, we obviously have

$$a_k \equiv 2a_{k-1} \bmod (N+k) \ \ \forall k \ge 1,$$

whence a_k will be even as soon as $N+k$ is $(k>0)$. Thus, $a_k=0$ (smallest k) implies $N+k$ even. It is also not difficult to see that we can restrict our attention to sequences with even $N = 2M$, so that in the n-tuple $(a_2, a_4, \ldots, a_{2n})$ the a_{2i} are even integers in the interval $[0, N+2i)$. If they would behave like "random," the probability that **none** of them equals 0 is easy to compute.

Indeed, the total number of n-tuples $(b_1, b_2, \ldots, b_n)$ with $b_i \in N \cap [0, 2M+2i)$, b_i even, is equal to the product $(M+1)\cdot(M+2)\cdot\cdots\cdot(M+n)$, while the total number of such n-tuples where no b_i is zero equals $M\cdot(M+1)\cdot\cdots\cdot(M+n-1)$. Thus, the chance for such an n-tuple not to contain 0 is

$$\frac{M(M+1)\cdot(M+n-1)}{(M+1)(M+2)\cdots(M+n)} = \frac{M}{M+n}.$$

Clearly, this number tends to 0 if n tends to infinity.

We include a small table in which the reader may find some numerical results concerning the "randomness" of the a_i.

N	$\bar{l}_N$	$\bar{l}'_N$
100	925.9	693.9
200	5902.3	2016.5
300	9999.3	2307.2
500	9993.6	8802.7
1000	10610.8	5701.3
2000	7389.8	50789.5
4000	11885.0	69030.1
8000	55131.3	95802.9

Here, $\bar{l}_N$ is the arithmetic mean of the numbers l_α that are defined as the smallest number k for which a_k is 0 $(\alpha = 1, 2, \ldots, N-1)$.

The number $\bar{l}'_N$ is the arithmetic mean of 1000 numbers l'_N, which has the same meaning as the l_i but where the a_k are chosen at random in $[0, N+k)$. Note that the l'_i will vary from one time to another. The reader who wishes to verify these numbers will probably not find the same ones.

3. SOME NONSTANDARD EXPANSION OF NUMBERS

First, note that a necessary condition for the sequences $s(N,\alpha)$ to end in a string of zeros is that α is a *rational number* with denominator of the form 2^t, for some $t \in \mathbb{N}$. Indeed, the equality $a_k = 0$ (for some $k \in \mathbb{N}_0$) implies $a_k = 2a_{k-1} - N - k$, which means that a_{k-1} is a rational number with denominator 2. From this it follows immediately that a_{k-2} must be a rational number with denominator 4. Continuing this proves our assertion.

In what follows, we shall discuss an "expansion of real numbers" that is (in some way) similar to what is known as "binary expansion."

Theorem 1: Every real number α in the interval $[0, 2]$ can be written as an infinite sum

$$\alpha = \sum_1^\infty \delta_k \frac{k}{2^k}, \quad \text{where the } \delta_k \text{ are 0 or 1.} \tag{2}$$

This is a special case of a more general theorem of Brown [1] that reads as follows:

If $\{r_i\}$ is a non-increasing sequence of real numbers with $\lim r_i = 0$ and $\{k_i\}$ is an arbitrary sequence of positive integers then every real number x in the interval $[0, \Sigma_{i=1}^{\infty} k_i r_i]$ can be expanded in the form $x = \Sigma_{i=1}^{\infty} \beta_i r_i$, where the β_i are integers satisfying $0 \le \beta_i \le k_i$, for all i, if and only if $r_p \le \Sigma_{i=p+1}^{\infty} k_i r_i$ for all $p \ge 1$.

The reader may verify that the conditions of this theorem are fulfilled when $r_i = i/2^i$, $k_i = 1$. However, to see the connection with the sequences mentioned in the introduction, it will be convenient to give a proof of this particular case.

Before doing so, notice that when the sum in (2) is *finite*, α will be a rational number with denominator of the form 2^t, $t \in \mathbb{N}_0$. About the converse, we state

Conjecture 2: Every rational number whose denominator is a power of 2 has a finite expansion (2).

We shall see that Conjecture 1 implies Conjecture 2. This implies that our numerical investigations provide a proof for the fact that *every rational number in* $[0, 2]$ *whose denominator is* 2^t, $t \le 2\,000\,000$, *can be expanded as a finite sum* (2).

Proof of Theorem 1: Let us abbreviate the numbers $k/2^k$ as u_k. First, note that the series $\Sigma_1^{\infty} u_k$ converges to 2. This follows from the equality

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad (x \in [0, 1))$$

which gives, after differentiation and multiplication by x,

$$x \cdot \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k.$$

Taking $x = 1/2$ gives the desired result

It is also clear from this that any series of the form (2) converges.

Now let α be an arbitrary element of $(0, 2)$ (the case $\alpha = 0$ or 2 is trivial). We define the numbers δ_k and the numbers B_k as follows:

If $\alpha \ge u_1$ $(= 1/2)$, then $\delta_1 = 1$, else $\delta_1 = 0$; $B_1 = \alpha - \delta_1 u_1$.

If $B_1 \ge u_2$ $(= 2/2^2)$, then $\delta_2 = 1$, else $\delta_2 = 0$; $B_2 = B_1 - \delta_2 u_2$.

If $B_2 \ge u_3$ $(= 3/2^3)$, then $\delta_3 = 1$, else $\delta_3 = 0$; $B_3 = B_2 - \delta_3 u_3$.

...

Our algorithm produces the digits δ_k by a so-called greedy expansion.

It suffices to show that the sequence $(B_1, B_2, \ldots)$ has limit 0. To do so, put

$$a_0 = \alpha,$$
$$a_k = 2^k B_k \quad (k = 1, 2, \ldots).$$

Then it is clear that we have $a_{k+1} = 2a_k - \delta_{k+1}(k+1)$, whence, by the definition of the δ_k:

$$a_{k+1} = \begin{cases} 2a_k & \text{if } < k+1, \\ 2a_k - (k+1) & \text{otherwise.} \end{cases}$$

Since, previously, we noted that every a_k is in $[0, k+2)$, we have $B_k \in [0, \frac{k+2}{2^k}]$, which completes the proof.

Note also that if **one** of the numbers B_i is zero, then so are all B_j when $j \geq i$; note further that the expansion is not "unique." To see this, define numbers η_k and numbers B'_k in the following way:

If $\alpha > u_1$ $(= 1/2)$, then $\eta_1 = 1$, else $\eta_1 = 0$; $B'_1 = \alpha - \eta_1 u_1$,

If $B'_1 > u_2$ $(= 2/2^2)$, then $\eta_2 = 0$; $B'_2 = B'_1 - \eta_2 u_2$,

If $B'_2 > u_3$ $(= 3/2^3)$, then $\eta_3 = 0$; $B'_3 = B'_2 - \eta_3 u_3$,

thus constructing a sequence B'_j of real numbers none of which will ever be zero.

The corresponding numbers a'_k $(= 2^k B_k)$ then satisfy a slightly different recursion, namely,

$$a'_{k+1} = \begin{cases} 2a'_k & \text{if } \leq k+1, \\ 2a'_k - (k+1) & \text{otherwise,} \end{cases}$$

so that in this case a'_{k+1} might be in the interval $(0, k+3]$ ... .

The proof of the theorem leads to the construction of a sequence $s(N, a)$ with $a = \alpha$ and $N = 0$ as defined in the introduction.

Now, suppose $\alpha \in [0, 2]$ is a rational number of the form $k/2^m$ with $k, m \in \mathbb{N}_0$. Then a_i is a rational number with denominator 2^{m-i} $(i = 1, 2, \ldots, m)$ and will be an integer for $i > m$. From the proof, it is also clear that a_i is in the interval $[0, 2+i)$. It is also easy to see that a_i is in the interval $[0, i)$ when $i > m+1$.

Thus, to see if every such α has a **finite** expression (2), it suffices to see if every series $s(N, a)$ with $N < m$ and a an integer in the range $1, 2, \ldots, N-1$ will "end" in zeros. We took $N =$ 2 000 000 and found $a_K = 0$ for some $K \leq 4\,588\,298\,126$ (the computations took several hours on a fast PC).

Since the expansion (2) is not unique, it is possible that Conjecture 2 is true even if Conjecture 1 should prove false.

4. OTHER ANALOGS WITH BINARY EXPANSIONS

There is another analog of the expansion (2) with "binary expansions." Consider a number α such that the δ_k are *periodic*, i.e., there exists a nonzero natural number p such that

$$\delta_k = \delta_{p+k} \tag{3}$$

for all $k \in \mathbb{N}$. In such a case, we have

$$\alpha = \sum_{i=1}^{p} \delta_i \frac{i}{2^i} + \sum_{i=1}^{p} \delta_i \frac{p+i}{2^{p+i}} + \sum_{i=1}^{p} \delta_i \frac{2p+i}{2^{2p+i}} + \cdots = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{p} \delta_i \frac{kp+i}{2^{kp+i}} \right).$$

Theorem 2: α is a rational number.

Proof: Define the polynomial $v(x)$ as $\sum_{i=1}^{p} \delta_i x^i$ and the real function $\varphi(x)$ as $\sum_{i=1}^{\infty} \delta_i x^i$. By the periodicity of the δ_i, we have

$$\varphi(x) = v(x) + x^p v(x) + x^{2p} v(x) + \cdots = v(x)\frac{1}{1-x^p}.$$

Differentiation and multiplying with x gives

$$x\varphi'(x) = \sum_{i=1}^{\infty} \delta_i i x^i = \frac{x(1-x^p)v'(x) + v(x)px^p}{(1-x^p)^2}.$$

Putting $x = 1/2$ yields

$$\alpha = \frac{v'(1/2)2^{p-1}}{2^p - 1} + \frac{p2^p v(1/2)}{(2^p-1)^2}. \tag{4}$$

It is easy to see that the numerators of the two fractions are both integers, which proves the theorem.

As to nonpure periodic expansions [this happens when the relations (3) are true for all $k \geq$ some m], it is not difficult to show that these α differ from some purely periodic number by a rational number with 2^m in the denominator.

It should be noted also that the pure periodic expansion of a number is in general **not** the one obtained by the greedy procedure explained in the proof of Theorem 1. For instance, if we take α to be 8/9, the sequence (δ_k) in Theorem 1 would be the nonpure periodic

$$(1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots)$$

while we have the equality

$$\frac{8}{9} = \sum_{k=1}^{\infty} \frac{2k}{2^{2k}}.$$

which is obviously pure periodic.

The converse of Theorem 2, however, is not true. This is seen by examining the second fraction in (4). Its numerator equals $\delta_1 2^{p-1} + \delta_2 2^{p-2} + \cdots + \delta_p$, which can be **any** of the values 0, 1, 2, 3, ..., 2^{p-1}. However, this is not sufficient to cancel enough factors of the denominator to yield **any** prescribed denominator. For instance, the number 1/3 is never equal to any periodic expression (2).

This may be considered as a (weak) argument that Conjecture 1 could fail to be true.

Remark: The number 2 plays a special role in all of the preceding in the following way. Consider series of the form

$$\sum_{k=1}^{\infty} \delta_k k c^k \quad (c \in [0, 1)) \tag{5}$$

where the δ_k are 0 or 1.

It is clear that such a series converges to a number of the interval $[0, A]$

$$A = \sum_{k=1}^{\infty} kc^k = \frac{c}{(1-c)^2}.$$

Using these notations, we can prove

Theorem 3: Every real number in the interval $[0, A]$ can be expressed as a sum (5) if and only if $c \in [1/2, 1)$.

The proof is an immediate consequence of Brown's theorem applied to this case. Indeed, Brown's theorem states that every real number in the interval $[0, A]$ is expressible as a sum (5) if and only if

$$pc^p \leq \sum_{k=p+1}^{\infty} kc^k = c^p\left(\sum_{i=1}^{\infty}(pc^i + ic^i)\right) = c^p\left(\frac{pc}{1-c} + \frac{c}{(1-c)^2}\right), \ \forall p \in \mathbb{N}.$$

This is equivalent to $p(2c-1)(c-1) \leq c, \ \forall p \in \mathbb{N}$, and this holds if and only if $c \in [1/2, 1)$. Q.E.D.

Therefore, extensions of our results when c is of the form $1/n, n \in \mathbb{N}, n > 2$ are not very likely to hold.

It is worthwhile to note that the number 2 has a similar role when looking at expansions of real numbers in the form

$$\sum_{k=1}^{\infty} \delta_k c^k \quad (c \in [0, 1)), \text{ where the } \delta_k \text{ are 0 or 1}$$

(which includes binary expansions). In the same way as above, we obtain

Theorem 4: Every real number in the interval $[0, A]$ can be expressed as a sum $\sum_{k=1}^{\infty} \delta_k c^k$ if and only if $c \in [1/2, 1)$.

This theorem has a surprising geometric interpretation. Consider for every infinite string

$$\delta = (\delta_1, \delta_2, \delta_3, \ldots); \ \delta_i = 0 \text{ or } 1$$

a real function $\varphi_\delta(x)$ defined by

$$\varphi_\delta(x) = \sum_{k=1}^{\infty} \delta_k x^k, \ x \in [0, 1).$$

Clearly, one has $0 \leq \varphi_\delta(x) \leq \frac{x}{1-x}, \ \forall x \in [0, 1)$. Now, by Theorem 4, every point of the unbounded set $\{(a, b) | 0.5 \leq a < 1; \ 0 \leq b \leq \frac{a}{1-a}\}$ belongs to at least one curve $y = \varphi_\delta(x)$, while some points of the bounded region $\{(a, b) | 0 \leq a < 0.5; \ 0 \leq b \leq \frac{a}{1-a}\}$ may fail to lie on any such curve. An example of such a point is $(1/3, \lambda)$, where λ is a positive real number less than 0.5, whole ternary expansion contains a two.

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