AVERAGE NUMBER OF NODES IN BINARY DECISION DIAGRAMS OF FIBONACCI FUNCTIONS*

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1. INTRODUCTION

A binary decision diagram (BDD) is a directed graph representation of a switching function $f(x_1, x_2, \ldots, x_n)$. Subfunctions of f correspond to nodes in the BDD; f itself is represented by a source node, i.e., a node with no incoming arcs. Attached to this node are two outgoing arcs, labeled 0 and 1, that go to descendent nodes representing $f(x_1, x_2, \ldots, 0)$ and $f(x_1, x_2, \ldots, 1)$, respectively. Attached to each of these nodes are descendent nodes, where x_{n-1} is replaced by 0 and 1, etc. This process is repeated until all variables are assigned values. The last assigned functions are a constant **0** and **1**, which correspond to sink nodes, i.e., nodes with no outgoing arcs. If two nodes represent the same function, they are merged into one node, and if the descendents of one node η are the same, η is removed. If $f = 1$ (0) for some assignment of values to $x_1, x_2, \ldots,$ and x_n, then there is a path in the BDD for f from the source node to the sink node **1** (**0**) for that assignment. Figure 1(a) shows the BDD of the OR function on four variables. As is usual, the arrows are omitted; all arcs are assumed to be directed down. As can be seen, there is a path from the source node to the node labeled **1** if and only if at least one variable is 1. Figure 1(b) shows the BDD of the AND function of four variables, which is the mirror image of the OR function BDD.

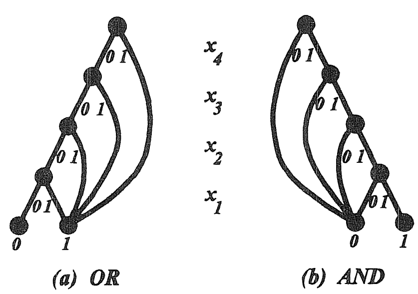

FIGURE 1. BDD's of the OR and AND Function on Four Variables

There is significant work on this topic dating back to 1959 [5]. In spite of this, there are few enumerations of nodes in BDD's of useful classes of functions. Symmetric functions, which are unchanged by a permutation of variables, have received some attention. The worst case number of nodes is known [3], [6], [7], as well as the average number of nodes [1].

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We demonstrate another class of functions and characterize its BDD. A *threshold function*, $f(x_1, x_2, \ldots, x_n)$, has the property that $f = 1$ if and only if $w_n x_n + w_{n-1}x_{n-1} + \cdots + w_1 x_1 \geq T$, where w_i and T are integers and the logic values, 0 and 1, of x_i are viewed as integers. The value of $w_n x_n + w_{n-1}x_{n-1} + \cdots + w_1 x_1$, for some assignment of values to $x_1, x_2, \ldots$, and x_n, is called the *weighted sum*. A threshold function is completely specified by a *weight-threshold* vector $(w_n, w_{n-1}, \ldots, w_1; T)$. For example, the four-variable OR and AND functions have weight-threshold vectors $(1,1,1,1; T)$, where $T = 1$ and 4, respectively. A *Fibonacci function* is a threshold function with weight-threshold vector $(F_n, F_{n-1}, \ldots, F_2, F_1; T)$, where F_i is the i^{th} Fibonacci number and $0 < T < F_{n+2}$. For example, the Fibonacci functions associated with weight-threshold vectors $(3, 2, 1, 1; 1)$ and $(3, 2, 1, 1; 7)$ correspond to the OR and AND function, respectively. on four variables. The *BDD of a Fibonacci function* is a BDD in which a path from the source node to a sink node is a sequence of arcs associated with variables of descending weights. Figure 2 shows the BDD's of all of the other four-variable Fibonacci functions, which have a weight-threshold vector $(3, 2, 1, 1; T)$, for $1 < T < 7$; thus, Figures 1 and 2 represent the entire set of seven four-variable Fibonacci function BDD's.

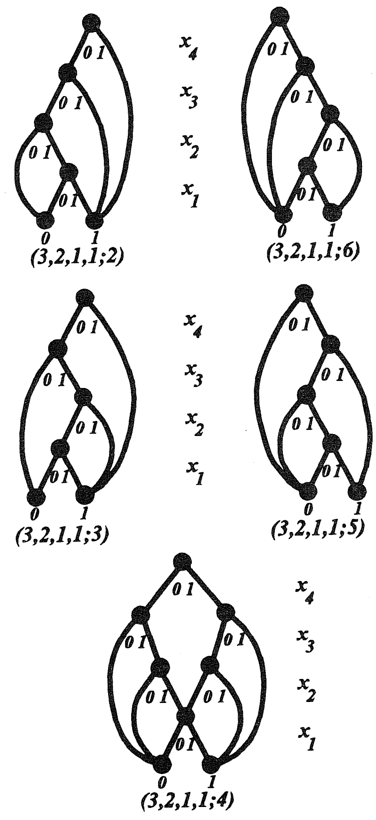

FIGURE 2. BDD's of Other Fibonacci Functions on Four Variables

The representation of a Fibonacci function by a BDD is related to the representation of integers by the Fibonacci number system, for which there exist many papers (see, e.g., [2], [4]). That is, every positive integer N can be represented as $N = \alpha_n F_n + \cdots + \alpha_2 F_2 + \alpha_1 F_1$, where F_i is a

Fibonacci number and $\alpha_i \in \{0, 1\}$. In a BDD, there is a path from the source node to **1** for all assignments of values to α_i, for $0 \le i \le n$, such that $N \ge T$.

2. STRUCTURE OF THE BDD'S OF FIBONACCI FUNCTIONS

In preparation for the calculations of the average number and variance of nodes in BDD's of Fibonacci functions, we consider the structure of such BDD's. Figure 3 shows how the structure near the source node depends on the threshold. Specifically, it shows that the destination of arcs emanating from the source node depends on the value of x_n. Figure 3a shows the *Type a* structure. As shown, if $0 < T \le F_n$, the arc corresponding to $x_n = 1$ goes to **1**. That is, for this range of T and this value of x_n, the weighted sum exceeds or equals the threshold, and $f = 1$. If $x_n = 0$, then the weighted sum exceeds or equals the threshold if and only if the Fibonacci function corresponding to the weight-threshold vector $(F_{n-1}, F_{n-2}, \ldots, F_1; T)$ is 1. The latter is represented by a node that is the 0 descendent of the source node.

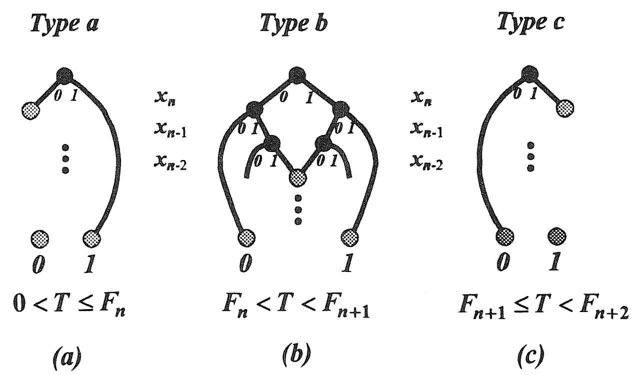

FIGURE 3. Structure of the BDD of a Fibonacci Function

A similar analysis of the case $F_{n+1} \le T < F_{n+2}$, which corresponds to a *Type c* structure, shows that there is mirror image symmetry with a Type a structure, as can be seen by comparing Figure 3(c) with 3(a).

Consider the remaining values of T, which correspond to a *Type b* structure. Figure 3(b) shows that, for this structure, both $x_n = 0$ and $x_n = 1$ yield nodes at the next lower level. If $x_n x_{n-1} = 11$, the weighted sum is at least $F_n + F_{n-1} = F_{n+1}$, and this equals or exceeds the threshold regardless of the values of the remaining variables. Thus, there is a path from the source node to **1** for $x_n x_{n-1} = 11$. If $x_n x_{n-1} = 00$, the weighted sum can be no greater than $F_{n-2} + F_{n-3} + \cdots + F_1 = F_n - 1$. Thus, the threshold is neither equaled nor exceeded, and there is a path from the source node to **0**. If $x_n x_{n-1} = 01$, the weighted sum ranges from a minimum of F_{n-1} to a maximum of $F_{n-1} + F_{n-2} + \cdots + F_1 = F_{n+1} - 1$, for which $f = 0$ and 1, respectively. It follows that there is a path from the source node to a non-sink node corresponding to $x_n x_{n-1} = 01$. A similar analysis shows that there is a non-sink node corresponding to $x_n x_{n-1} = 10$. Similarly, non-sink nodes exist for $x_n x_{n-1} x_{n-2} = 011$ and for $x_n x_{n-1} x_{n-2} = 100$. Indeed, since $F_{n-1} + F_{n-2} = F_n$, the weights are the same for the last two cases, and they correspond to the same node.

A fourth type of structure, the *Type d* structure, consists of a node that has as descendents the two sink nodes **0** and **1**. This represents the Fibonacci function with weight-threshold vector

$(1; 1)$. Indeed, all threshold functions contain this structure. As can be seen in Figures 1 and 2, it is part of all BDD's of Fibonacci functions on four variables.

Composing BDD's of Fibonacci Functions

Consider combining structures. If a BDD has a Type a structure, as shown in Figure 3(a), and the weight-threshold vector associated with the Fibonacci function of the source node is $(F_n, F_{n-1}, \ldots, F_1; T)$, where $0 < T \leq F_n$, then the node that is the 0 descendent of the source node corresponds to a Fibonacci function with weight-threshold vector $(F_{n-1}, F_{n-2}, \ldots, F_1; T)$. Further, the 0 descendent can also have a Type a structure, in which case the node at $x_n x_{n-1} = 00$ is associated with the weight-threshold vector $(F_{n-2}, F_{n-3}, \ldots, F_1; T)$. Indeed, this process can be repeated until the last variable, which has a Type d structure. Represent this composition as $a^i d$, for $i \geq 1$, and the set of all such compositions as aa^*d. Here, $a^* = \{\lambda, a, aa, aaa, \ldots\}$, where λ is the *null* structure. Thus, aa^*d represents the concatenation of one or more Type a structures followed by a Type d structure. By this convention, the right to left sequence in the string representation corresponds to the top to bottom sequence in the BDD. Such compositions occur only when $T = 1$, which is the OR function. For example, the BDD in Figure 1(a) is described by $a^3 d$ and corresponds to the weight-threshold vector $(3, 2, 1, 1; 1)$.

In a similar manner, repeated use of the Type c structure corresponds to a BDD described by $c^i d$, for $i \geq 1$, producing a mirror image of $a^i d$. Such compositions occur only when $T = F_{n+2} - 1$, which is the AND function. For example, the BDD in Figure 1(b) is described by $c^3 d$ and corresponds to the weight-threshold vector $(3, 2, 1, 1, 7)$.

Consider combining Types a and c. For example, let the source node have a Type a structure and its 0 descendent have a Type c structure. Thus, the 0 descendent of the source node is associated with weight-threshold vector $(F_{n-1}, F_{n-2}, \ldots, F_1; T_1)$, where $0 < T_1 = T \leq F_n$. But, because it is a Type c structure, we have $F_n \leq T_1 < F_{n+1}$. Since there is only one value of T_1 that satisfies both inequalities, it follows that $T = T_1 = F_n$. It follows that the weight-threshold vector of the 1 descendent of the 0 descendent of the source node is $(F_{n-2}, F_{n-3}, \ldots, F_1; F_{n-2})$, since $F_{n-2} = F_n - F_{n-1}$. Thus, this node has a Type a structure whose 0 descendent has a Type c structure, etc., until all variables are exhausted. The resulting compositions are described by $ac(ac)^*(a+\lambda)d$, where + is set union. A similar result occurs if the source node has a Type c structure, in which case the resulting compositions are described by $ca(ca)^*(\lambda + c)d$. These observations have important implications in the composition of the BDD's of Fibonacci functions.

- A BDD can consist of a sequence of one or more Type a structures followed by an alternating sequence of Type c and Type a structures, as described by $a^*(ca)^*(\lambda + c)d$. Similarly, a BDD can consist of a sequence of one or more Type c structures followed by an alternating sequence of Type a and Type c structures, as described by $c^*(ac)^*(a+\lambda)d$. As an example, see the BDD's in Figures 1 and 2 corresponding to thresholds $T = 1, 2, 3, 5, 6$, and 7.
- A "crest" pattern of the form shown in Figure 3(b) can only occur after a sequence of Type a structures exclusively or Type c structures exclusively. On the contrary, if both types occur, we have a situation as described immediately above, in which case, no crest can occur anywhere in the BDD.

Consider the composition of the BDD's of Fibonacci functions involving the crest pattern; i.e., Type b structures. Figure 4 shows how the BDD structure depends on T in the range $F_n < T < F_{n+1}$. Here, the top node of the crest pattern is the source node of the BDD. It is interesting how the structure changes at the boundary between ranges and that Fibonacci numbers define these boundaries. In the BDD for $T = F_n + F_{n-3}$ and $F_n + F_{n-2}$, the bottom node of the crest corresponds to a weight-threshold vector where the threshold is F_{n-3} and F_{n-2}, respectively. From the discussion above, this part of the BDD consists of a sequence of structures chosen alternatively as Type a and Type c. Again, the mirror image symmetry of the BDD's of Fibonacci functions is evident.

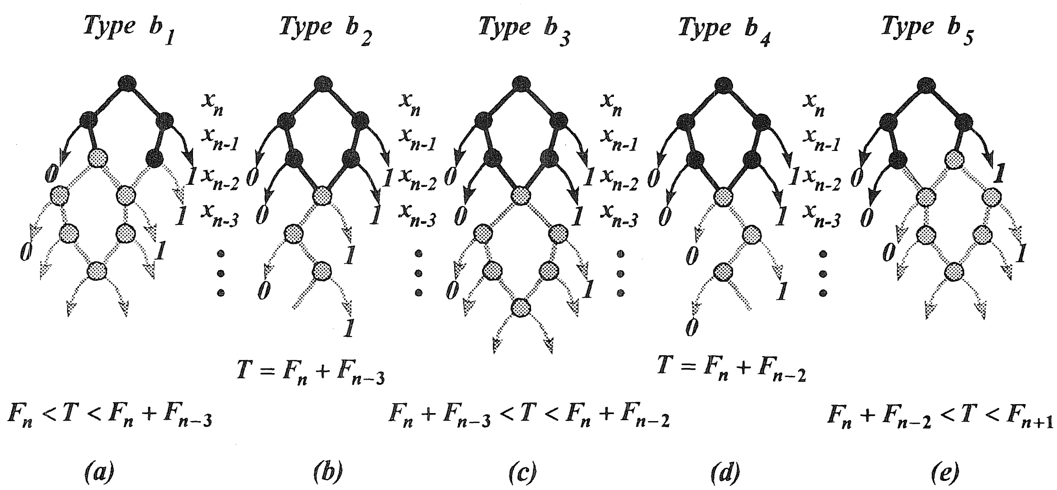

FIGURE 4. Structure of the BDD of a Fibonacci Function in the Range $F_n < T < F_{n+1}$

3. THE AVERAGE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

Let $T(x, y)$ be the ordinary generating function for the number of BDD's of Fibonacci functions, where x tracks the number of variables and y tracks the number of nodes. Let $t_{n,i}$ be the number of BDD's of n-variable Fibonacci functions that have i nodes. From the results in the previous section, it follows that if $t_{n,i} > 0$, then $i \geq n+2$, since there is at least one node for every variable and two sink nodes **0** and **1**. Thus, a term in $T(x, y)$ is

$$T(x, y) = \cdots + x^n(t_{n,n+2}y^{n+2} + t_{n,n+4}y^{n+4} + \cdots) + \cdots. \tag{1}$$

Note that $t_{n,n+2i+1} = 0$ for $i = 1, 2, \ldots,$ since additional nodes beyond the minimum number $n+2$ occur because each crest pattern contributes two *additional* nodes to the node count. If we differentiate (1) with respect to y and set y equal to 1, the resulting coefficient of x^n is the total number of nodes in all BDD's of Fibonacci functions on n variables. Dividing by the number of BDD's of such functions yields the average number of nodes.

To derive $T(x, y)$, we use the classification given in Figure 3. That is,

$$T(x, y) = T_a(x, y) + T_b(x, y) + T_c(x, y) + xy^3, \tag{2}$$

where $T_a(x, y)$, $T_b(x, y)$, and $T_c(x, y)$ are the generating functions for Type a, b, and c structures, respectively, and xy^3 is the generating function for the Type d structure. By symmetry,

$$T_c(x, y) = T_a(x, y). \tag{3}$$

We can derive $T_a(x, y)$ by observing that there are two types of BDD's counted in $T_a(x, y)$— those that contain at least one crest pattern (but not at the very top, which are Type b structures) and those that do not. BDD's of the first type are enumerated by $x^i y^i T_b(x, y)$ for $i \geq 1$. Recall that the top crest pattern is preceded by a sequence of Type a structures. BDD's of the second type are enumerated by $ix^{i+1}y^{i+3}$ for $i \geq 1$. That is, this type of structure consists of a sequence of i Type a structures ending with a Type d structure or followed by Type c structures alternating with Type a structures ending with a Type d structure. The string representation for this is $a^i(ca)^*(\lambda + c)d$. The factor x^{i+1} counts the variables involved, and the factor y^{i+3} counts the nodes involved, including the two sink nodes **0** and **1**. Therefore,

$$T_a(x, y) = xyT_b(x, y) + x^2y^2T_b(x, y) + \cdots + x^i y^i T_b(x, y)$$
$$+ \cdots + x^2y^4 + 2x^3y^5 + \cdots + ix^{i+1}y^{i+3} + \cdots,$$

which can be written as

$$T_a(x, y) = \frac{xy}{1 - xy} T_b(x, y) + \frac{x^2y^4}{(1 - xy)^2}. \tag{4}$$

We can calculate $T_b(x, y)$ by observing that BDD's of Fibonacci functions containing a crest at the source node can be completed in three ways. Figure 4(c) shows that the bottom node of the crest is the top node of a Type b structure. The number of ways to choose a Type b structure is counted by the generating function $T_b(x, y)$. The contribution of the crest itself to the variable and node count is expressed as x^3y^5. Thus, the total contribution to the variable and node count is expressed as $x^3y^5T_b(x, y)$. Figures 4(b) and 4(d) show that the bottom node can also be the source node of a BDD with one node per variable expressed as $(ac)^*(a + \lambda)d$ and $(ca)^*(\lambda + c)d$, respectively. The contribution of these nodes is expressed as $2x^2y^4 + 2x^3y^5 + 2x^4y^6 + \cdots$. The coefficient 2 occurs because of the two ways this part of the BDD can occur [Figures 4(b) and 4(d)]. The superscript of x counts variables and the superscript of y counts nodes, including the two sink nodes **0** and **1**. The generating function for this power series is $2x^2y^4/(1 - xy)$. A sub-BDD consisting of just the lowest variable and the three nodes, including two sink nodes **0** and **1** (i.e., a Type d structure) should also be included, and this is expressed as xy^3. Figures 4(a) and 4(e) show that more than one crest can also be cascaded so that each adjacent pair of crests share an arc and two nodes. In this case, the top BDD contributes two variables and four nodes. Since there are two ways for this to happen, the contribution is described by $2x^2y^4T_b(x, y)$. Considering all three ways to form a Type b BDD, we have

$$T_b(x, y) = x^3y^5\left[T_b(x, y) + xy^3 + \frac{2x^2y^4}{1 - xy}\right] + 2x^2y^4T_b(x, y). \tag{5}$$

Solving for $T_b(x, y)$ in (5) yields

$$T_b(x, y) = \frac{x^4y^8(1 + xy)}{(1 - xy)(1 - x^3y^5 - 2x^2y^4)}. \tag{6}$$

From (2), (3), (4), and (6), we can write

$$T(x, y) = \frac{x^3y^5 - 2x^3y^7 + xy^3}{(1-xy)^2(1-x^3y^5-2x^2y^4)}. \tag{7}$$

Recall that a typical term in (7) is given in (1). We can find the total number of nodes by differentiating (7) with respect to y and setting y to 1. Doing this yields

$$T(x) = \frac{3+2x}{(1-x-x^2)^2} - \frac{7+3x}{1-x-x^2} + \frac{1}{(1-x)^2} + \frac{3}{1-x}. \tag{8}$$

The number of n-variable BDD's is calculated as follows. There are as many BDD's as there are integer threshold functions from 1 to the largest threshold. The largest threshold is the same as the largest weighted sum, $1+1+2+3+\cdots+F_n = F_{n+2}-1$. Note that we exclude BDD's corresponding to $T=0$ and F_{n+2}, which are trivial. Therefore, the average number of nodes is the coefficient t_n of the power series expansion of (8) divided by $F_{n+2}-1$. Table 1 shows the average number of nodes as calculated in this way.

TABLE 1. The Average Number of Nodes in BDD's of Fibonacci Functions of n Variables

Number of Variables n	Average Number of Nodes	Standard Deviation on the Number of Nodes
1	3.000	0.000
2	4.000	0.000
3	5.000	0.000
4	6.286	0.700
5	7.667	0.943
6	9.200	1.327
7	10.818	1.585
8	12.519	1.853
9	14.273	2.049
10	16.070	2.224
11	17.897	2.354
12	19.745	2.462
13	21.608	2.543
14	23.481	2.609
15	25.361	2.659
∞	$1.8944\,n$	$0.2540\sqrt{n}$

Asymptotic Approximation

Consider now the average number of nodes in BDD's of Fibonacci functions when the number of variables is large. We can factor the quadratic denominators in the partial fraction expansion (8), forming a partial fraction expansion in which denominators involve linear factors only. That is, we can rewrite (8) as

$$T(x)=\frac{\frac{11+5\sqrt{5}}{10}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^2}-\frac{\frac{61+31\sqrt{5}}{10\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)}+\cdots, \tag{9}$$

where $\cdots$ represents terms whose contributions to t_n, the coefficient of x_n in the power series expansion of $T(x)$, are negligible for large n compared to the contributions from the terms shown. Specifically, missing terms have denominators that are powers of $(1+(2/\sqrt{5}+1)x)$ and $(1+x)$. Indeed, the second term in (9) is negligible for large n compared to the first term; we include it for a reason that will become clear in the next section. The contribution to t_n from these terms is

$$\left(\frac{11+5\sqrt{5}}{10}(n+1)-\frac{61+31\sqrt{5}}{10\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}-1}\right)^n. \tag{10}$$

The number of BDD's of n-variable Fibonacci functions, $F_{n+2}-1$, is approximated by

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}, \tag{11}$$

when n is large. Dividing (10) by (11) yields the following asymptotic approximation to the average number of nodes in BDD's of Fibonacci functions on n variables,

$$\frac{5+2\sqrt{5}}{5}n-\frac{2+6\sqrt{5}}{5}\approx 1.8944n-3.0832, \tag{12}$$

which is asymptotic to $1.8944n$, for large n. As can be seen from Table 1, 3.0832 is significant for the values of n shown here.

4. THE VARIANCE OF THE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

We can calculate the variance on the number of nodes in BDD's of Fibonacci functions using the generating function for the distribution of nodes given in (7). That is, if X is a random variable, then the variance $\sigma^2(X)$ of X is given as

$$\sigma^2(X)=E(X^2)-E^2(X),$$

where $E(X^2)$ is the expected value of X^2 and $E(X)$ is the expected value of X. $E(X)$ was calculated in the previous section. $E(X^2)$ can be calculated by differentiating (7) with respect to y, multiplying by y, differentiating with respect to y again, and setting y to 1. In the resulting expression, the coefficient of x^n is ΣX^2. Dividing this by the number of BDD's of Fibonacci functions yields $E(X^2)$. Differentiating (7) with respect to y, multiplying by y, differentiating with respect to y again, and setting y to 1 yields

$$\frac{16+10x}{(1-x-x^2)^3}-\frac{49+16x}{(1-x-x^2)^2}+\frac{49+25x}{1-x-x^2}+\frac{6}{(1-x)^3}-\frac{1}{(1-x)^2}-\frac{23}{1-x}+\frac{2}{1+x}. \tag{13}$$

The coefficient of x^n in the power series expansion of (13) is decreased by $E^2(X)$ and the result divided by the number of BDD's of Fibonacci functions on n variables, $F_{n+2}-1$, to get the variance on the number of nodes for n-variable BDD's of Fibonacci functions. This yields $\sigma^2(X)$. Table 1 shows the standard deviation, $\sigma(X)$, of the number of nodes, as calculated in this way.

Asymptotic Approximation

Consider the standard deviation on the number of nodes in BDD's of Fibonacci functions when the number of variables is large. We can rewrite (13) as

$$\frac{\frac{47+21\sqrt{5}}{5\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^3}-\frac{\frac{691+277\sqrt{5}}{50}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)^2}+\frac{\frac{2131+881\sqrt{5}}{50\sqrt{5}}}{\left(1-\frac{2}{\sqrt{5}-1}x\right)}+\cdots, \tag{14}$$

where the contribution to ΣX^2 for large n from other terms is negligible compared to the contribution from the terms shown. The contribution of these three terms is indeed

$$\left[\frac{47+21\sqrt{5}}{10\sqrt{5}}(n^2+3n+2)-\frac{691+277\sqrt{5}}{50}(n+1)+\frac{2131+881\sqrt{5}}{50\sqrt{5}}\right]\left[\frac{2}{\sqrt{5}-1}\right]^n. \tag{15}$$

Dividing this result by (11), an approximation to the number of BDD's of Fibonacci functions, yields $E(X^2)$. Subtracting from this the square of the average number of BDD's of Fibonacci functions, as given in (12), yields the following asymptotic approximation to the variance on the number of nodes in BDD's of Fibonacci functions

$$\frac{100-44\sqrt{5}}{25}n+\frac{228-28\sqrt{5}}{25}\approx 0.0645n+6.6156. \tag{16}$$

Note that there is no n^2 term in (16); the n^2 term in $E(X^2)$ has been canceled by an identical term in $E^2(X)$. Therefore, terms of order n are needed in the asymptotic expressions for $E(X^2)$ and $E^2(X)$. This is why we included in (10) and (12) an asymptotically insignificant term.

Equation (16) is an expression for $\sigma^2(X)$. The standard deviation $\sigma(X)$ is then

$$\sqrt{\frac{100-44\sqrt{5}}{25}n+\frac{228-28\sqrt{5}}{25}}\approx\sqrt{0.0645n+6.6156}, \tag{17}$$

which is asymptotic to $0.2540\sqrt{n}$, for large n. As can be seen from Table 1, 6.6156 is significant for the values of n shown.

5. DISTRIBUTION OF THE NUMBER OF NODES IN BDD'S OF FIBONACCI FUNCTIONS

Figure 5 shows the distribution of nodes in the BDD's of Fibonacci functions, as computed from (7). Here, the number of variables and the number of nodes in BDD's are plotted horizontally, while the number of Fibonacci functions is plotted vertically. A vertical line represents the number of Fibonacci functions whose BDD's have the number of variables and the number of nodes as specified by the coordinates in the horizontal plane. The vertical axis shows the *log* of the number of functions. Note the linear increase in the log of number of functions with the number of nodes in BDD's for a fixed number of variables, which corresponds to an exponential increase in the number of functions.

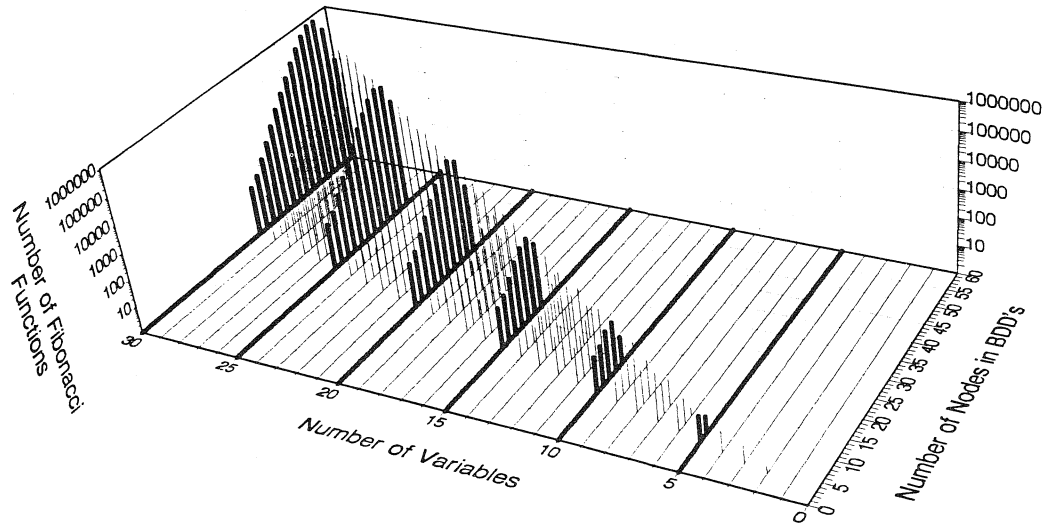

FIGURE 5. Distribution of Fibonacci Functions by Nodes and Variables

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